IMAT - 64

DISCRETE MATHEMATICS

UNIT - 1

Preposition:

A statement (or) preposition is a decloration sentense that is either true (or) false but not both.

Example: (i) the earth is drawn

(ii) $2+3=5$

Negation:

If p is a statement the negation of p is the statement ~~the~~ not p denoted by $\sim p$

Touth table of $\sim p$

p	$\sim p$
T	F
F	T

Conjuction:

If p and q are statement they conjuction of p and q is the compound statement p and q which is denoted by $p \wedge q$

Touth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example :

Find the conjuction of p and q

p = it is snowing

q = I am Cold.

$p \wedge q$: It is snowing and I am Cold.

Disjuction :

If p and q are statement then the disjuction of p and q is compound statement p (or) q is denoted by $p \vee q$.

Truth table $p \vee q$:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example :

p : 2 is a possitive integer

q : $\sqrt{2}$ is a rational number

Find $p \vee q$.

$p \vee q$: 2 is a possitive Integers (or) $\sqrt{2}$ is rational numbers.

Example :-

1) Find a truth table for $(p \wedge q) \vee (\sim p)$.

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \vee (\sim p)$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

2) $\sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

3. truth table for $(\sim p \vee q) \wedge r$

p	q	r	$\sim p$	$\sim p \vee q$	$(\sim p \vee q) \wedge r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	F

H.W.
4) $(p \vee q) \vee r$.

p	q	r	$p \vee q$	$(p \vee q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

5) $\sim p \wedge (q \vee r)$

p	q	r	$\sim p$	$q \vee r$	$\sim p \wedge (q \vee r)$
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	F	T	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	F	F

6) $p \wedge (\sim p \vee \sim r)$.

6/12/2019 Find the truth table for (i) $(r \wedge \sim q) \vee (p \vee r)$.

(i)

p	q	r	$\sim q$	$r \wedge \sim q$	$p \vee r$	$(r \wedge \sim q) \vee (p \vee r)$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	T	F	F	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	F	F

(ii) $(q \wedge r) \wedge (p \vee \sim r)$

p	q	r	$\sim r$	$q \wedge r$	$p \vee \sim r$	$(q \wedge r) \wedge (p \vee \sim r)$
T	T	T	F	T	T	T
T	T	F	T	F	T	F
T	F	T	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	T	F	T	F	T	F
F	F	T	F	F	F	F
F	F	F	T	F	T	F

The universal Qualification:

The universal Qualification of a predicate $p(x)$ is a statement for all values of x $p(x)$ is true.

Example:

If $p(x) = -(-x)$

Then $p(x)$ is positive for all x. x is real number.

Existential Quantification :-

The existential quantification of a predicate $p(x)$ is a statement there exist a value of x for which $p(x)$ is true

Let $p(x) = x + 1 < 4$

Then $p(x)$ is true only for $x = 1, 2$.

$\therefore p(x)$ is an existential quantification.

2.2. Conditional statements:

Implication: $(\Rightarrow)$

If p and q are statement the compound statements if p then q denoted as $p \Rightarrow q$ is called a conditional statement or implication.

Example:

From $p \Rightarrow q$ if p: I am hungry

q: I will eat

$p \Rightarrow q$: If I am hungry then I will eat.

Truth table:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition: Contrapositive:

The Contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

Example:

p: It is raining

q: I get wet

$\sim p$: It is not raining

$\sim q$: I am not get wet.

$p \Rightarrow q$: If it is raining, then I get wet

$\sim q \Rightarrow \sim q$: If I am not wet, then it is not raining.

By condition :

If p and q are statements then the compound statements p if and only if q denoted as $\Leftrightarrow$ is called an equivalance or By condition.

Example : $p : 3>2$

$q : 3-2>0$

$p \Leftrightarrow q : 3>2$ iff $3-2>0$.

Truth table :

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Compute a truth table of the statement $(p \Rightarrow q) \Longleftrightarrow (\sim q \Rightarrow \sim p)$.

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim q \Rightarrow \sim p$	$(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	F	T

13/2/2019

1 Tautology :
A statement that is true for all possible values is called a tautology

Eg : $(p \wedge (p \Rightarrow q)) \Rightarrow q$.

2. Contradiction (or) absurdity
A statement that is always false is called a contradiction (or) absurdity.

3. Contingency:
A statement that can be either true or false contingency.

1. Example for tautology: $(P \wedge (P \Rightarrow q)) \Rightarrow q$.

P	q	$P \Rightarrow q$	$P \wedge (P \Rightarrow q)$	$P \wedge (P \Rightarrow q) \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

2. Example for absurdity: $(P \wedge \sim P)$

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

3. Example for contingency: $(P \Rightarrow q) \wedge (P \vee q)$.

P	q	$P \rightarrow q$	$P \vee q$	$(P \Rightarrow q) \wedge (P \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

4. Equivalent condition: $(\equiv)$. [Commutative property)
Example:

S.T $(p \vee q) \equiv q \vee p$.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

i.e the conditional statement satisfies the commutative property.

Similarly we can prove $(p \wedge q) \equiv (q \wedge p)$

5. Associative property:

Example: i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$.

ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

i)

p	q	r	$q \vee r$	$p \vee q$	$p \vee (q \vee r)$	$(p \vee q) \vee r$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	F	T	T
F	F	F	F	F	F	F

Similarly we can prove $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

It follows associative property.

6. Distributive property:

Example: i) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

ii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

i)

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

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Similarly we can prove $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

It follows distributive property.

Idempotent property:

Example: (i) $p \wedge p \equiv p$

(ii) $p \vee p \equiv p$.

(i)

p	p	p ∨ p
T	T	T
F	F	F

p	~p	~(~p)
T	F	T
F	T	F

Property of Negation:

(i) $p \equiv \sim(\sim p)$

(ii) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$ (iii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

Example: Each of the following is a tautology:

(i) $(p \wedge q) \Rightarrow q$

(ii) $p \Rightarrow (p \vee q)$

(iii) $q \Rightarrow (p \vee q)$

(iv) $(\sim p \wedge (p \vee q)) \Rightarrow q$.

(v) $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

Property of Negation:

Example:

(ii) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

p	q	~p	~q	p ∨ q	~(p ∨ q)	(~p) ∧ (~q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

equal

(iii) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

p	q	~p	~q	p ∧ q	~(p ∧ q)	(~p) ∨ (~q)
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(i) $(p \wedge q) \Rightarrow q$.

p	q	p ∧ q	(p ∧ q) ⇒ q.
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Example for tautology: (i) $p \Rightarrow (p \vee q)$

p	q	$p \vee q$	$p \Rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(ii) $q \Rightarrow (p \vee q)$

p	q	$p \vee q$	$q \Rightarrow p \vee q$.
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(iv) $(\sim p \wedge (p \vee q)) \Rightarrow q$.

p	q	$\sim p$	$(p \vee q)$	$\sim p \wedge (p \vee q)$	$(\sim p \wedge (p \vee q)) \Rightarrow q$.
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

(v) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow p \Rightarrow r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

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(M) / Con p/n

2.3. Methods of Proof:

THEOREM:

If n be a integer PT if n^2 is odd then n is odd.

Proof:

p : n^2 is odd

q : n is odd

Then we have to prove $p \Rightarrow q$.

Since $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

we Prove this theorem by Contrapositive

$\sim p$: n^2 is even

$\sim q$: n is even

If $\sim q$: n is even

ie. $n = 2m$ where m is an integer

Now $n^2 = (2m)(2m) = 4m^2 = 2(2m^2)$ = even number

n^2 is even number

$= \sim p$

$\therefore \sim q \Rightarrow \sim p$

Theorem:

Prove there is no rational number p/q whose square is 2. In otherwords $\sqrt{2}$ is an irrational number.

Proof:

We prove this theorem by Contradiction.

Suppose there exists a rational number p/q

whose square is 2. where p and q have no common factors.

$$(p/q)^2 = 2$$

$\Rightarrow p^2 = 2q^2 \rightarrow ①$ = even number

$\Rightarrow p$ is an even number

$p = 2n$ where n is an integer

i.e $\Rightarrow p^2 = 4n^2 \rightarrow ②$

② in ① $\Rightarrow 4n^2 = 2q^2$

$\Rightarrow 2n^2 = q^2$ = even number

$\therefore q$ is also even number

$\therefore$ Both p & q are even.

$\therefore p$ & q have common factors 2

which is a contradiction to p & q have no common factors

$\therefore$ no rational number p/q whose square is 2.

Example:

1/12/2019 Let m and n be integers. P.T. $n^2 = m^2$ iff n is m (or) n is $-m$.

Proof:

Suppose $n^2 = m^2$

Taking square root on both sides.

$n = \pm m$

i.e, $n = m$ (or) $n = -m$.

Conversely, suppose $n = m$ (or) $n = -m$

If $n = m$

Squaring $\Rightarrow n^2 = m^2$.

If $n = -m$

Squaring $\Rightarrow n^2 = m^2$ $\therefore$ For both cases $n^2 = m^2$ //

Ex:
Prove or disprove the statement that if x and y are real numbers $(x^2 = y^2)$ then $x = y$

We disprove this statement

If $x^2 = y^2 = 9$

Then $x = 3$ and $y = -3$

i.e, $x \neq y$

2.4. Mathematical induction:

Suppose the statement to be proved in the form $\forall n \geq n_0 . P(n_0)$, where n_0 is a fixed integer.

Suppose, we wish to show that $P(n)$ is true for all integers $n \geq n_0$

Then (a), $P(n_0)$ is true

(b) If $P(k)$ is true for $k \geq n_0$.

Then $P(k+1)$ must also be true.

Example:

Show by induction method for all $n \geq 1$.

$$1 + 2 + 3 + \dots\dots\dots + n = \frac{n(n+1)}{2}.$$

Proof:

BASIC STEP:

$P(n)$ is $1 + 2 + \dots + n = \frac{n(n+1)}{2}$.

L.H.S of $P(1) = 1$

R.H.S of $P(1) = \frac{1(2)}{2} = 1$.

$\therefore$ L.H.S = R.H.S

$\therefore P(1)$ is true

INDUCTION STEP:

Suppose $P(k)$ is true

i.e., $1+2+3+\dots+k = \frac{k(k+1)}{2}$

Now If $n = k+1$

$$1+2+3+\dots+k+k+1 = \frac{k(k+1)}{2} + k+1$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$P(k+1)$ is true

Let $A_1, A_2, \dots A_n$ be n sets. Show by mathematical induction $\left(\overline{\bigcup_{i=1}^{n} A_i}\right) = \bigcap_{i=1}^{n} \overline{A_i}$ for $n \geq 1$

Proof:

$P(n)$ is $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$

BASIC STEP:

If $n=1$

$L.H.S = \overline{A_1}$

$R.H.S = \overline{A_1}$

$\therefore L.H.S = R.H.S.$

$\therefore$ It is true

INDUCTION STEP:

Suppose $P(n)$ is true for $n=k$

i.e., $\overline{\bigcup_{i=1}^{k} A_i} = \bigcap_{i=1}^{k} \overline{A_i}$

i.e., $\overline{\bigcup_{i=1}^{k} A_i} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}$

i.e., $\overline{A_1 \cup A_2 \cup \dots A_k} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}$

Now if $n = k+1$

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \overline{A_1 \cup A_2 \cup \ldots A_k \cup A_{k+1}}$$

$$= \overline{A_1 \cup A_2 \cup \ldots \ldots A_k} \cap (\overline{A_{k+1}})$$ [By De Morgan's law & $\overline{A \cup B} = \overline{A} \cap \overline{B}$]

$$= \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \ldots \cap \overline{A_{k+1}}$$

$$= \bigcap_{i=1}^{k+1} \overline{A_i}$$

$\therefore P(k+1)$ is true

19/12/2019.

Prove by induction method $\forall n \geq 1, n! \geq 2^{n-1}$

Proof:

$P(n)$ is $n! \geq 2^{n-1}$

BASIC STEP:

L.H.S of $P(1) = 1! = 1$.

R.H.S of $P(1) = 2^0 = 1$

$\therefore$ L.H.S = R.H.S.

INDUCTION STEP:

Suppose $P(n)$ is true for k steps

i.e., $k! \geq 2^{k-1}$

Now,

$$(k+1)! = 1 \cdot 2 \cdot \ldots \cdot k(k+1)$$

$$= k!\,(k+1)$$

$$\geq 2^{k-1}\,(k+1)$$

$$\geq 2^{k-1} \cdot 2 \cdot \quad (\because k \geq 1)$$

$$= 2^k.$$

$\therefore (k+1)! \geq 2^k$

$\therefore P(k+1)$ is true

H.W

Prove by Induction method (i) $2+4+6+\ldots+2n = n(n+1)$

(ii) $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$

(iii) $1+2+2^2+\ldots\ldots+2^n = 2^{n+1}-1$

(iv) $1^2+2^2+3^2+\ldots\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

(i) Proof :

$P(n)$ is $2+4+6+\ldots+2n = n(n+1)$

BASIC STEP :

L.H.S of $P(1) = 2$

R.H.S of $P(1) = 1(1+1) = 2$.

$\therefore$ L.H.S = R.H.S.

INDUCTION STEP :

Suppose $P(n)$ is true for k steps.

i.e., $2+4+6+\ldots\ldots+2k = k(k+1)$

Now

$$\underbrace{2+4+6+\ldots+2k}_{k(k+1)}+2(k+1) = k(k+1)+2(k+1)$$

$$= k(k+1)+2(k+1)$$

$$= (k+1)(k+2)$$

$P(k+1)$ is true.

(ii) Proof :

$P(n)$ is $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}$.

BASIC STEP :

L.H.S of $P(1) = 1$

R.H.S of $P(1) = 1$

$\therefore$ L.H.S = R.H.S.

INDUCTION STEP :

Suppose $P(n)$ is true for k steps.

i.e., $1^2+3^2+5^2+\ldots+(2k-1)^2 = \frac{k(2k+1)(2k-1)}{3}$.

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$$1^2+3^2+5^2+\dots+(2k-1)^2+(2k+1)^2 =$$

$$\frac{k(2k+1)(2k-1)}{3} + (2k+1)^2$$

$$= \frac{k(2k+1)(2k-1)+3(2k+1)^2}{3}$$

$$= \frac{(2k+1)\left[k(2k-1)+3(2k+1)\right]}{3}$$

$$= \frac{2k+1}{3}\left[2k^2-k+6k+3\right] = \frac{2k+1}{3}\left[2k^2+5k+3\right]$$

$$= \frac{2k+1}{3}\left[(k+1)(2k+3)\right]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$\therefore$ P(k+1) is true

4. Proof:

$P(n)$ is $1^2+2^2+3^2+ \dots +n^2 = \frac{n(n+1)(2n+1)}{6}$.

BASIC STEP:

L.H.S of $P(1) = 1^2 = 1$

R.H.S of $P(1) = \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$

L.H.S = R.H.S.

INDUCTION STEP:

Suppose $p(n)$ is true for k step

i.e., $1^2+2^2+3^2+ \dots +k^2 = \frac{k(k+1)(2k+1)}{6}$

Now,

$$1^2+2^2+3^2+ \dots +k^2 +(k+1)^2 = \frac{k(k+1)(2k+1)}{6} +(k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right].$$

$$= \frac{(k+1)}{6}\left[2k^2+k+6k+6\right].$$

$$= \frac{(k+1)}{6}\left[2k^2+7k+6\right].$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}.$$

$\therefore P(k+1)$ is true.

3. Proof:

$P(n)$ is $1+2+2^2+ \dots +2^n = 2^{n+1}-1 \quad n \geq 0$.

BASIC STEP:

L.H.S of $P(0) = 2^0 = 1$.

R.H.S of $P(0) = 2^{1}-1 = 1$

$\therefore$ L.H.S = R.H.S.

3. Proof:

INDUCTION STEP:

Suppose P(n) is true for k step.

i.e., $1+2+2^2+\cdots+2^k = 2^{k+1}-1$

Now,

$$1+2+2^2+\cdots+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}$$

$$= 2(2^{k+1})-1$$

$$= 2^{k+2}-1$$

$\therefore$ P(k+1) is true.

2/1/2020

5. $5+10+15+\cdots+5n = \dfrac{5n(n+1)}{2}$.

$\dfrac{5(k+1)(k+2)}{2}$

BASIC STEP:

L.H.S of $p(1) = 5$

R.H.S of $p(1) = \dfrac{5(1+1)}{2} = 5$

L.H.S = R.H.S.

INDUCTION STEP:

Suppose P(n) is true for k step

i.e., $5+10+15+\cdots+5k = \dfrac{5k(k+1)}{2}$.

Now

$$5+10+15+\cdots+5k+5(k+1) = \frac{5k(k+1)}{2}+5(k+1)$$

$$= \frac{5k(k+1)+10(k+1)}{2}$$

$$= \frac{(k+1)(5k+10)}{2}$$

$$= \frac{(k+1)\,5\,(k+2)}{2}$$

$\therefore$ P(k+1) is true.

6. $1+a+a^2+\cdots+a^{n-1} = \dfrac{a^n-1}{a-1}$. $\dfrac{a^{k+1}-1}{a-1}$

BASIC STEP : $P(n) = 1+a+a^2+\cdots+a^{n-1} = \dfrac{a^n-1}{a-1}$

L.H.S of $P(1) = 1$

R.H.S of $P(1) = \dfrac{a-1}{a-1} = 1$.

L.H.S = R.H.S.

INDUCTION STEP:

Suppose $P(n)$ is true for k step

i.e., $1+a+a^2+\cdots+a^{k-1} = \dfrac{a^k-1}{a-1}$

Now

$$1+a+a^2+\cdots+a^{k-1}+a^k = \frac{a^k-1}{a-1}+a^k.$$

$$= \frac{a^k-1+(a-1)a^k}{a-1}$$

$$= \frac{\cancel{a^k}-1+a^{k+1}-\cancel{a^k}}{a-1}$$

$$= \frac{a^{k+1}-1}{a-1}$$

$\therefore P(k+1)$ is true.

2. $a+ar+ar^2+\cdots+ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$ $\dfrac{a(1-r^{k+1})}{1-r}$

BASIC STEP:

$P(n) = a+ar+ar^2+\cdots+ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$.

L.H.S of $P(1) = ar^0 = a$.

R.H.S of $P(1) = \dfrac{a(1-r)}{1-r} = a$.

$\therefore$ L.H.S = R.H.S.

INDUCTION STEP:

Suppose P(n) is true for k step.

i.e.,

$$a + ar + ar^2 + \cdots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$$

Now,

$$a + ar + ar^2 + \cdots + ar^{k-1} + ar^{(k)} =$$

$$\frac{a(1-r^k)}{1-r} + ar^k.$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a - ar^k + ar^k - ar^{k+1}}{1-r}$$

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$$= \frac{a(1-r^{k+1})}{1-r}.$$

$\therefore$ P(k+1) is true.

Prove by induction $\left(\bigcup_{i=1}^{n} A_i\right) \cap B = \bigcup_{i=1}^{n} (A_i \cap B)$.

Proof:

BASIC STEP: If $n=1$, L.H.S $= A_1 \cap B$

R.H.S $= A_1 \cap B$.

L.H.S = R.H.S

It is true.

INDUCTION STEP:

If it is true for k step.

$$\left(\bigcup_{i=1}^{k} A_i\right) \cap B = \bigcup_{i=1}^{k} (A_i \cap B).$$

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Now

$\cancel{\left(\bigcup_{i=1}^{k+1} A_i\right) \cap B = \bigcup_{i=1}^{k+1} (A_i \cap B)}$.

Now,

$$\left(\bigcup_{i=1}^{k+1} A_i\right) \cap B = \left(A_1 \cup A_2 \cup \ldots A_k \cup A_{k+1}\right) \cap B$$

$$= \left(\left(\bigcup_{i=1}^{k} A_i\right) \cup (A_{k+1})\right) \cap B.$$

$$= \left(\bigcup_{i=1}^{k} A_i\right) \cap B \cup (A_{k+1} \cap B)$$

$$= \bigcup_{i=1}^{k} (A_i \cap B) \cup (A_{k+1} \cap B)$$

$$= \bigcup_{i=1}^{k+1} (A_i \cap B).$$

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2.5 Mathematical statements

Eg: The sum of two odd integers is an even integers

Proof:

Let $(2k+1)$ and $(2n+1)$ are two odd integers

Now, $(2k+1) + (2n+1)$

$= 2k + 2n + 2$

$= 2(k+n+1)$

$= 2m$ where $k+n+1 = m$.

is an even integer. $\therefore$ Sum of the two odd integers is an even integer

Let S be the set of matrices of the form $\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix}$ where $a, b \in \mathbb{R}$. P.T (S, matrix multiplication) satisfies the commutative property.

Solution :

If $C = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix}$.

$D = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 \end{bmatrix}$.

Then

$$CD = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_1b_2 \end{bmatrix}$$

$$DC = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_1b_2 \end{bmatrix}$$

$\Rightarrow CD = DC$. $\therefore$ It satisfies commutative property.

2.6. Logic and problem solve :

Exact cover : Given a set A and a finite number of subsets of A, $A_1, A_2, \ldots A_n$. Then the collection S is called a exact cover of A if it satisfies the following properties

i) Any two in S are disjoint.

ii) The union of all sets in S is A

Example :

Let $A = \{a, b, c, d, e, f, g, h, i, j\}$.

$A_1 = \{a, c, d\}$, $A_2 = \{a, b, c, d\}$.

$A_3 = \{b, f, g\}$, $A_4 = \{d, h, i\}$

$A_5 = \{a, h, j\}$ $A_6 = \{e, h\}$

$A_7 = \{c, i, j\}$, $A_8 = \{i, j\}$.

Is there is an exact cover of A ?

solution :

$S = \{A_1, A_3, A_6, A_8, A_4, A_5, A_7\}$.

$\therefore \{A_1, A_3, A_6, A_8\}$. is an exact Cover of A

2. $U = \{1,2,3,4,5,6,7,8,9,10\}$

$A_1 = \{2,4,6,7\}$ $A_2 = \{1,2,3\}$, $A_3 = \{9,10\}$.

$A_4 = \{5,8,1\}$, $A_5 = \{1,3,5\}$ find exact cover of U.

~~$\therefore \{A_1, A_3, A_4, A_5\}$ is an exact cover of U.~~

$= \{2,4,6,7, 9,10, \underline{5}, 8, \underline{1}, \underline{1}, 3, \underline{5}\}$.

Because of the repetition of 1, 5 it not exact cover.

This is not exact Cover of U.

3.1 Permutation.

THEOREM: Suppose that two tasks T_1 and T_2. are to be performed in a sequence. If T_1 can be performed in n_1 ways and T_2 can be performed in n_2 ways then the sequence $T_1 T_2$ can be performed in $n_1 n_2$ ways.

Proof:

T_1 can be performed in n_1 ways

T_2 can be performed in n_2 ways, and so on, T_n can be performed in n_n ways.

$\therefore T_1 T_2$ can be performed in $n_1 n_2$ ways and $T_1 T_2 T_3$ can be performed in $n_1 n_2 n_3$ ways and so on.

∴ This is called multiplication of Principle of counting.

THEOREM 2:

Example: A label identifier for a computer system consists of one letter followed by three ~~diff~~ digits. If repeatitions are allowed how many distinct label identifiers are possible.

Proof:

Let T_1 be the task consists of first letters which can be performed by 26 ways.

Let T_2 be the task consist of first digit can be performed by 10 ways.

T_3 can be the task consists of 2nd digit can be performed by 10 ways.

T_4 can be the task of 3rd digit can be performed by 10 ways.

By multiplicative principle $T_1T_2T_3T_4$ can be

∴ By multiplicative principle

performed by $26 \times 10 \times 10 \times 10 = 26000$ ways.

Problem:

(a) How many different sequences each of length 'r' can be formed using elements from A if elements in the sequence may be repeated

(b) all the elements in the sequence must be distinct.

Proof (a) Let T_1 be the task pefomed by n ways. T_2 be the task performed by n ways

....

T_r be the task pefomed by n way

By multiplication principle

$T_1 T_2 \dots T_r$ can be performed by $n \cdot n \cdot \dots \cdot n$ (r times)

$= n^r$ (repeated)

(b) If T_1 be the task performed by n ways.

T_2 be the task pefomed by n-1 way

T_3 be the task pefomed by n-2 ways

⋮

T_r be the task pefomed by $n-(r-1)$ way.

$\therefore$ By multiplication principle

$T_1 T_2 \dots T_n$ Can be pefomed by

$n(n-1)(n-2) \dots (n-r+1)$

$n! = \underbrace{n(n-1)(n-2) \dots (n-r)+1}_{n!} \cdot (n-r) \dots 1$ $\quad nP_r = \frac{n!}{(n-r)!}$

$$nP_r = \frac{n!}{(n-r)!} \text{ (not repeated)}$$

Example:

How many 3 letter words can be famed from the letters in the {a, b, y, z} if repeated letters are allowed.

Solution: The number of 3 letter words famed in n^r

{aaa, aby, byz, yza, zab, ...}

$n = 4,\ r = 3.$

$\therefore n^r = 4^3 = 64$

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Permutation $nPr = \frac{n!}{(n-r)!}$

Example:

Let $A = \{1, 2, 3, 4\}$. Find the permutation for $r = 3$.

$$nPr = 4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4 \times 3 \times 2 \times 1$$

$$4P_3 = 24$$

Example:

How many words of three distinct letters can be formed from the letters of the word MAST

$n = 4 \quad r = 3$.

$4P_3 = 24$

24 words of three distinct letters can be formed from the word MAST

How many distinciable permutations of these letters in the word BANANA

$n = 6 \quad r = 6$

$$6P_6 = \frac{6!}{0!} = 6! = 720.$$

The number of distinciable (distinguishable) permutations that can be formed from a collection of n objects where the first objects appears k_1 times the second object appear k_2 times and so on. is

$$\frac{n!}{k_1! \, k_2! \ldots k_t!} \quad \text{where } k_1 + k_2 + \ldots + k_t = n$$

Example:

The number of distinguishable words that can be formed from the letters of MISSISSIPPI

Solution: $k_1 = 1$ (M) $k_2 = 4$ (I) $k_3 = 4$ (S) $k_4 = 2$

$+ \quad n = 11$

$\therefore$ The number of destinct words is $\frac{n!}{k_1!\,k_2!\,k_3!\,k_4!}$

$$= \frac{11!}{2!\,4!\,4!\,2!} \qquad = \frac{11\times10\times9\times8\times7\times6\times5\times4!}{2\times1\; 4!\; 4\times3\times2\times1\times2\times1}$$

$= 34,650.$

① A bank password consist of two letters from alphabet followed by two degit. How many different passwords are there.

② A coin is tossed four times & the result of each toss is recorded. How many different sequence of heads and tails are possible.

③ Find the number of different permutations of the letters in the word ~~group~~ GROUP.

④ Find the number of destinct able permutations of the letters in (associative) ASSOCIATIVE.

Solutions:

1. Let T_1 be the task consist of first letter which can be performed by 26 ways.

Let T_2 be the task consist of ~~first~~ 2nd letter which can be performed by 26 ways.

Let T_3 be the task consist of 3rd digit can be performed by 10 ways.

Let T_4 be the task consist of 4th digit can be performed by 10 ways.

By multiplication principle $T_1T_2T_3T_4$ can be performed by $26\times26\times10\times10 = 67600$ ways.

3.2

2. $n = 2 \quad r = 4$

$n^r = 2^4 = 16$ possibilities.

5. A die is tossed 4 times how many different sequence are there.

$n = 6 \quad r = 4.$

$n^r = 6^4 = 1296.$

6 $n = \{a, b, c, d, e, f\}$ and $r = 2$. Find the permutation.

$n = 6 \quad r = 2.$

$nPr = 6P_2 = \frac{6!}{4!\,2!} = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}} = 30$

4. ASSOCIATIVE:

$k_1 = 2 \quad k_2 = 2, \; k_3 = 1, \; k_4 = 1 \quad k_5 = 2, \; k_6 = 1$

$k_7 = 1 \quad k_8 = 1.$

$= \frac{11!}{2!\,2!\,1!\,1!\,2!\,1!\,1!} = \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 2 \times 1}$

$= 4989600$

3. GROUP.

$k_1 = 1 \quad k_2 = 1 \quad k_3 = 1 \quad k_4 = 1 \quad k_5 = 1$

$= \frac{5!}{1!\,1!\,1!\,1!\,1!} = 120.$

7. REQUIREMENTS.

$k_1 = 2, \; k_2 = 2, \; k_3 = 1, \; k_4 = 1, \; k_5 = 1, \; k_6 = 1$

$k_7 = 1 \quad k_8 = 1 \quad k_9 = 1 \quad = \frac{11!}{2!\,2!\,1!\,1!\,1!\,1!\,1!\,1!\,1!} = 9979200$

3.2 Combinations

THEOREM:

Let **A** be set with $|A| = n$, and $0 \le r \le n$.

Then the no. of combinations of the elements of A taken 'r' at a time.

i.e., the no. of r elements subsets of **A** is

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

Example:

Compute the no. of distinct five-card hands, that can be dealt from a deck of 52 cards.

Solution:

$$nC_r = \frac{n!}{r!\,(n-r)!} \qquad n = 52, \; r = 5$$

$$52C_5 = \frac{52!}{5!\,47!} = 2598960.$$

THEOREM:

Suppose k selections are to be made from n items with order and repeats are allowed then the no. of ways of selection is $(n+k-1)C_k$

example:

In howmany ways can a prize winner choose 3 CD's from the top 10 list if repeats are allowed.

Solution: $r = 3$ $n = 10$.

$$(n+r-1)C_r = (10+3-1)C_3 = 12C_3$$

$$= 220.$$

1 Compute the following (i) $^7C_7$ (ii) $^7C_4$

2 Compute $^nC_{n-1}$

3 S.T $^nC_r = {}^nC_{n-r}$.

(1) (i) $^7C_7 = 1$ (ii) $^7C_4 = 35$

$$= \frac{7!}{7!(7-7)!} \qquad \frac{7!}{4!(7-4)!} = \frac{7!}{4!\,3!}$$

$$= \frac{7!}{7!} = 1 \qquad = \frac{7\times 6\times 5\times 4!}{4!\,3\times 2\times 1} = 35$$

3. $^nC_r = \frac{n!}{r!(n-r)!} \rightarrow (1)$.

$$^nC_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,r!} \rightarrow (2).$$

From (1) & (2), $^nC_r = {}^nC_{n-r}$.

Example:

Suppose that a valide computer password consister of 7 character a first of which is a letter choose from the {A, B, C, D, E, F, G} and the remaining 6 character are teller from the English alphabet or digit how many different password are possible.

Let T_1 be the task can be performed by from a given set in $^7C_1$ ways = 7 ways

Let T_2 be the task can be performed from 26 letter and 10 digits in 36^6 ways

= 7 × 2,76,1782,336 ways.

$= 15,237,476,350$ ways.

How many different 7 person committe can be formed each containing 3 women from an available sets of 20 women and 4 men from an available set of 30 men.

Task 1 consist of 3 women from 20 women which can be performed by $20C_3$ ways

Task 2 consist of 4 men from 30 men which can be performed by $30C_4$ ways.

By multiplication principle

The total number of 7 person committee can be performed by $(20C_3)(30C_4)$ ways.

$$= \left(\frac{20!}{3!\,7!}\right)\left(\frac{30!}{4!\,26!}\right)$$

$$= 31241700.$$

20/1/2013 3.3 pigeonhole principle:

Theorem:

If n pigeons are assigned to m pigeonholes $m < n$ then atleast one pigeonhole contains two (or) more pigeons.

Proof:

Suppose each pigeonhole contains at most one pigeon. Then at most m pigeons are assigned but since $m < n$ not all pigeons have assigned in pigeonhole. This is an contradiction.

$\therefore$ At least one pigeonhole contains two (or) more pigeons.

Example:

S.T if any 5 numbers from one 1 to 8 are choosen then 2 of them will add to 9.

Solution: $A_1 = \{1, 8\}$, $A_2 = \{2, 7\}$, $A_3 = \{3, 6\}$, $A_4 = \{4, 5\}$ each of the 5 numbers choosen must belong to one these sets. Since there are four sets by pigeonhole principle 2 of the choosen belong to the same set.

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Extended pigeonhole principle:-

Theorem: If n pigeon are assigned to m pigeonholes then one the pigeonholes must contain atleast $\left\lceil \frac{n-1}{m} \right\rceil + 1$ pigeons.

Define: flooring number $\lfloor \ \rfloor$

$\lfloor 3.1 \rfloor = 3$.

$\lfloor -2.5 \rfloor = -3$.

Define: Ceiling number $\lceil \ \rceil$.

$\lceil 3.1 \rceil = 4$

$\lceil -2.5 \rceil = -2$

Proof:

Suppose each pigeonhole contains atmost $\left\lfloor \frac{n-1}{m} \right\rfloor$ pigeon.

Now $m \left\lfloor \frac{n-1}{m} \right\rfloor \leq m \left(\frac{n-1}{m} \right)$

$\leq (n-1)$.

which is a $\Rightarrow\Leftarrow$ to n pigeon assigned to m pigeonholes At least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons must be assigned in m pigeonholes.

Example:

S.T if any 30 peoples are selected then one way choose a subset of 5 show that all 5 where born on the same day of the week.

Proof: $n = 30$, $m = 7$

By using extend principle,

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{29}{7} \right\rfloor + 1 = \lfloor 4.1 \rfloor + 1 = 4 + 1$$

$$= 5$$

S.T if 30 dictionaries in a library contain a total of 61327 pages then one of the dictionaries must have at least 2045 pages.

Solution:

By using extened principle

$m = 30$

$n = 61327$

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{61327-1}{30} \right\rfloor + 1 = \left\lfloor \frac{61326}{30} \right\rfloor + 1$$

$$= \lfloor 2044.2 \rfloor + 1$$

$$= 2045$$

3.4. Elements of Probability :

Sample Space :

A set 'S' consisting of all the outcomes of the experiment is called a sample space.

Example : If a coin is tossed twice

Then sample space $S = \{HH, HT, TH, TT\}$.

Determine a sample space for an experiment of tossing a die twice.

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

Event :

A statement about the out come of an experiment will be either true or false is called an event.

Example :

an experiment consist of tossing a die twice determine the event.

(a) Sum of the number is 8

(b) Sum of the number is at least 10

(a) ⇒ $E = \{(2,6), (6,6), (3,5), (4,4), (5,3), (6,2)\}$

(b) $E = \{(4,6), (5,5), (6,4), (6,5), (6,6)\}$.

Certain event :

If A is a sample space of the experiment then A is itself of an event is called certain event and the empty set is an impossible event.

Example :

Consider the experiment of tossing the die and E be the event that the number is event F be the event that the number is odd. Find $E \cup F$, $E \cap F$ and $\bar{E}$.

Solution :

$S = \{1, 2, 3, 4, 5, 6\}$

$E = \{2, 4, 6\}$.

$F = \{1, 3, 5\}$.

$E \cup F = \{1, 2, 3, 4, 5, 6\}$.

$E \cap F = \{\phi\}$

$\bar{E} = \{1, 3, 5\}$.

$E \cup F$ is an certain event

$E \cap F$ is an impossible event.

Probability of the event :

23.1.20

To each event E has been assigned a number P(E) is called the probability of the event.

Axioms of probability :

(1) $0 \leq P(E) \leq 1$ for each event E.

(2) If A is a whole set $P(A) = 1$ & $P(\phi) = 0$

(3) If $E_1, E_2, \ldots E_n$ are mutually exclusive events. Then $P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$.

Example :

Consider the experiment as a sample space $A = \{1, 2, 3, 4, 5, 6\}$ and the elements of the probability have been assigned as follows.

$$A = \{1, 2, 3, 4, 5, 6\}.$$

$$P_1 = 1/12, \quad P_2 = 1/12, \quad P_3 = 1/3, \quad P_4 = 1/6.$$

$$P_5 = 1/4, \quad P_6 = 1/12.$$

Find the event "the outcome is an even number and compute P(E).

Solution :

$$E = \{2, 4, 6\}.$$

$$\therefore P(E) = P_2 + P_4 + P_6 = \frac{1}{12} + \frac{1}{6} + \frac{1}{12}.$$

$$= \frac{1+2+1}{12} = \frac{4}{12}$$

$$P(E) = \frac{4}{12} = \frac{1}{3}$$

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Equally likely outcomes :

$$P(E) = \frac{|E|}{|A|} = \frac{\text{no. of elements in the event } E}{\text{no. of elements in the sample space } A}.$$

Example :

1. Chose 4 ~~[illegible]~~ cards at random from a 52 cardsdeck what is the probability that 4 cards are king.

Solution :

The number of elements in these sample space is $52C_4 = \frac{52!}{4!(52-4)!} = \frac{52!}{4!\,48!}$

(En).

$= 270,725$.

The number of elements in the event E containing all 4 cards are kings is 1

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{1}{270,725}$$

$= 0.0000036\,94$.

2. A box contains. 6 red balls and 4 green balls. 4 balls are selected at radom from the box. What is the probability that two of the selected balls will be red and two of them will be green.

The no. of balls in these sample spacis is

$$10C_4 = \frac{10!}{4!(6!)} = 210.$$

Let T_1 be the task chooting 2 red balls from 6 red balls is $6C_2 = 15$

Let T_2 be the task chooting 2 green balls in $4C_2 = 18$.

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By multiplication principle

$\therefore$ the no. of elements in the event E containing T_1 and T_2 is ~~~~ $= 15 \times 6 = 90$ //.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{90}{210} = \frac{3}{7} //.$$

3. A die tossed three times and the resulting sequence is recorded. what is the probability of the event E that either all three numbers are equal or none of them is 4.

The no. of elements in the sample space S while tossing a die thrice is $n^r = 6^3 = 216$.

Let E be the event that contains all the elements are equal.

i.e., $E = \{(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)\}$.

$\therefore n(E) = 6$.

Let F be the event contains none of the value is 4 $n(F) = 5^3 = 125$

Also $E \cap F = \{(1,1,1) (2,2,2) (3,3,3), (5,5,5), (6,6,6)\}$.

$n(E \cap F) = 5$.

Now $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

$$= \frac{6}{216} + \frac{125}{216} - \frac{5}{216} = \frac{126}{216} = \frac{7}{12}$$

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A box contains 6 red balls and 4 green balls in which 4 balls are selected at random (i) If E is the event that no more than two balls are red. Compute the probability of E

(ii) If F is the event that no more than 3 balls are red. Compute the probability of F.

Solution:

Let S be the sample space taking 4 balls from 10 balls.

$$n(S) = {}^{10}C_4 = 210$$

(i) Given E be the event containing no more than 2 red balls.

$$E = E_0 \cup E_1 \cup E_2.$$

where E_0 is the event containing no red balls.

$$n(E_0) = {}^4C_4 = 1.$$

E_1 is the event containing 1 red ball & 3 green balls

$$n(E_1) = ({}^6C_1)({}^4C_3) = 24.$$

E_2 is the event containing 2 red ball & 2 green balls

$$n(E_2) = ({}^6C_2)({}^4C_2) = 90.$$

$$n(E) = n(E_0) + n(E_1) + n(E_2) = 1 + 24 + 90$$

$$= 115.$$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{115}{210} = \frac{23}{42}$$

(ii) Given F be the event containing no more than 3 red balls.

$$F = F_0 \cup F_1 \cup F_2 \cup F_3.$$

where F_0 is the event containing no red balls

$$n(F_0) = {}^4C_4 = 1.$$

F_1 is the event containing 1 red ball & 3 green balls

$$n(F_1) = ({}^6C_1)({}^4C_3) = 24.$$

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F_2 is the event containing 2 red balls and 2 green balls.

$$n(F_2) = (6C_2)(4C_2) = 90$$

F_3 is the event containing 3 red balls and 1 green balls.

$$n(F_3) = (6C_3)(4C_1) = 80$$

$$n(F) = n(F_0) + n(F_1) + n(F_2) + n(F_3)$$

$$= 1 + 24 + 90 + 80$$

$$= 195$$

$$\therefore P(F) = \frac{n(F)}{n(S)} = \frac{195}{210} = \frac{13}{14}.$$

1. $S = \{1,2,3,4,5,6\}$ be the sample space of experiment let $E = \{1,3,4,5\}$. $F = \{2,3\}$. $G = \{4\}$. Find the event $E \cup F$, $E \cap F$, $\bar{E} \cup F$, $\bar{F} \cap G$.

$S = \{1,2,3,4,5,6\}$, $F = \{2,3\}$

$E = \{1,3,4,5\}$, $\bar{F} = \{1,4,5,6\}$

$\bar{E} = \{2,6\}$, $G = \{4\}$.

$E \cup F = \{1,2,3,4,5\}$, $E \cap F = \{3\}$.

$\bar{E} \cap F = \{2\}$, $\bar{F} \cap G = \{4\}$.

$\bar{E} \cup F = \{2,3,6\}$.

2. Consider an experiment with sample space $S = \{1,2,3,4\}$ for which $P_1 = 2/7$, $P_2 = 3/7$ $P_3 = 1/7$ $P_4 = 1/7$. Find the probability of $E = \{1,2\}$, $F = \{1,3,4\}$

$E = \{1, 2\} = P_1 + P_2 = \frac{2}{7} + \frac{3}{7} = \frac{5}{7}$

$F = \{1, 3, 4\} = P_1 + P_3 + P_4 = \frac{2}{7} + \frac{1}{7} + \frac{1}{7} = \frac{4}{7}$.

Suppose that 3 balls are selected at random from an urn containing 7 red balls and 5 black balls. Compute the probability that (i) all 3 balls are red. (ii) at least 2 balls are black (iii) atmost 2 balls are black (iv) at least 1 ball is red. $n(S) = 12C_3$

(i) $7C_3$ (i) all 3 balls are red. $\rightarrow 7C_3$

(ii) at least 2 balls are black $\rightarrow E = E_1 + E_2$

$(5C_2)(7C_1) + (5C_3)$

(iii) atmost 2 balls are black $\rightarrow E = E_0 + E_1 + E_2$.

$(7C_3) + (5C_1)(7C_2) + (5C_2)(7C_1)$

(iv) at least 1 ball is red $\rightarrow E = E_1 + E_2 + E_3$.

$(7C_1)(5C_2) + (7C_2)(5C_1) + (7C_3)$

$n(S) = 12C_3 = 220$

(i) Given E be the event containing all 3 balls are red

$n(E) = 7C_3 = 35$

$P(E) = \frac{n(E)}{n(S)} = \frac{35}{220} = \frac{7}{44}$

(ii) Given E be the event containing at least 2 balls are black.

$E = E_1 + E_2$

where E_1 is the event containing 2 black balls and one red balls.

$n(E_1) = (5C_2)(7C_1) = 10 \times 7 = 70$

where E_2 is the event containing 3 balls are black.

$n(E_2) = 5C_3 = 10$.

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$n(E) = n(E_1) + n(E_2) = 70 + 10 = 80$

$$P(E) = \frac{n(E)}{n(S)} = \frac{80}{220} = \frac{4}{11} //$$

(iii) Given E be the event containing at most 2 balls are black.

$n(E) = n(E_0) + n(E_1) + n(E_2)$.

where E_0 is the event containg 3 black balls.

$n(E_0) = 7C_3 = 35$

where E_1 is the event containg 1 black ball and 2 red balls.

$n(E_1) = (5C_1)(7C_2) = (5)(21) = 105$.

where E_2 is the event containing 2 red balls and 1 black ball.

$n(E_2) = (5C_2)(7C_1) = (10)(7) = 70$.

$n(E) = n(E_0) + n(E_1) + n(E_2) = 35 + 105 + 70$

$= 210$.

$$P(E) = \frac{n(E)}{n(S)} = \frac{210}{220} = \frac{21}{22} //$$

(iv) Given E be the event containing at least 1 ball is red

$n(E) = n(E_1) + n(E_2) + n(E_3)$

where E_1 is the event containing 1 red and 2 black balls

$n(E_1) = 7C_1 \cdot 5C_2 = (7)(10) = 70$

where E_2 is the event containg 2 red and 1 black ball.

$n(E_2) = (7C_2)(5C_1) = 21 \times 5 = 105$.

where E_3 is the event containing 3 red balls.

$n(E_3) = 7C_3 = 35$.

$n(E) = 70 + 105 + 35 = 210$

$$P(E) = \frac{n(E)}{n(S)} = \frac{210}{220} = \frac{21}{22} //$$

2. Suppose 2 dices are tossed and the numbers recorded. What is the probability that (i) 4 was tossed (ii) prime number was tossed (iii) Sum of numbers is less than 5 (iv) Sum of numbers is atleast 7.

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

$n(S) = 36$.

(i) 4 was tossed:

Let the event be A

$A = \{(1,4), (2,4), (3,4), (4,4), (5,4), (6,4)$
$(4,1), (4,2), (4,3), (4,5), (4,6)\}$.

$n(A) = 11 \quad n(S) = 36$

$P(A) = \frac{n(A)}{n(S)} = \frac{11}{36}$

(ii) prime number was tossed:

Let the event be B.

$B = \{(1,1), (1,2), (1,3), (1,5), (2,1), (2,3), (2,5),$
$(3,1), (3,2), (3,3), (3,5), (5,1), (5,2), (5,3),$
$(5,5)\} (2,2)\}$

$n(S) = 16 \quad n(B) = 36$

$P(B) = \frac{16}{36} = \frac{8}{18}$

(iii) Sum of numbers is less than 5.

Let the event be C

$C = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\}$

$n(C) = 6 \quad n(S) = 36$

$P(C) = 6/36 = 1/6$

(iv) Sum of numbers is atleast 7.

Let the event be D

$n(D) = n(S) = 36$

$P(D) = \frac{21}{36}$

$D = \{(1,6), (2,5), (2,6), (3,4), (3,5), (3,6), (4,3), (4,4)$
$(4,5), (4,6), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3),$
$(6,4), (6,5), (6,6)\}$

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3.5 Recurrence Relation:

1. Eg: The recurrence Relation $a_n = a_{n-1} + 3$ with $a_1 = 4$. find the sequence

Solution:

$a_1 = 4$

$a_2 = a_1 + 3 = 4 + 3 = 7$

$a_3 = a_2 + 3 = 7 + 3 = 10$ and so on.

$\therefore$ The sequence is $4, 7, 10, \ldots$

2. $f_n = f_{n-1} + f_{n-2}$, $f_1 = f_2 = 1$ define fibonacci sequence

Solution:

$f_1 = 1$, $f_2 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$.

$f_4 = f_3 + f_2 = 2 + 1 = 3$.

$f_5 = f_4 + f_3 = 3 + 2 = 5$ and so on.

fibonacci sequence is $1, 1, 2, 3, 5, 8, \ldots$

Define:

One technique for finding an explicit formula for the sequence defined by recurrence relation is back tracking.

Eg: The recurrence relation is $a_n = a_{n-1} + 3$ with $a_1 = 2$ define the sequence $2, 5, 8, \ldots$. Find the explicit formula using back tracking.

Solution:

The recurrence relation is $a_n = a_{n-1} + 3$.

$$a_{n-1} = (a_{n-2} + 3) = a_{n-2} + 1 \cdot 3$$

$$a_n = (a_{n-2} + 3) + 3 = a_{n-2} + 2 \cdot 3.$$

$= (a_{n-3} + 3) + 2.3 = (a_{n-3} + 3.3)$

$= (a_{n-4} + 3) + 3.3 = (a_{n-4} + 4.3)$ and so on

At least $a_n = a_{n-(n-1)} + (n-1)3$.

$a_n = a_1 + (n-1)3$.

$= 2 + (n-1)3$

$= 2 + 3n - 3$

$= 3n - 1$ is the explicit formula.

Backtrack

2. To find a explicit formula for the sequence define by recurrence relation $b_n = 2b_{n-1} + 1$ with initial condition $b_1 = 7$

Solution: $b_n = 2b_{n-1} + 1$

$b_n = 2(2 b_{n-2} + 1) + 1$

$b_n = 2^3 b_{n-3} + 2^2 + 2 + 1$.

and so on.

$= 2^{n-1} b_{n-(n-1)} + 2^{(n-2)} + 2^{(n-1)} + \dots + 2 + 1$

$= 2^{n-1} b_1 + (2^{n-2} + 2^{n-1} + \dots 2 + 1)$

$= 2^{n-1} b_1 + 2^{n-1} - 1$

$= 7(2^{n-1}) + 1(2^{n-1}) - 1$

$= 8(2^{n-1}) - 1$

$= 2^3(2^{n-1}) - 1 \quad = 2^{n+2} - 1$

The explicit formula is $a_n = 2^{n+2} - 1$

$\therefore$ The sequence is 7, 15, 31, 63, ...

Theorem :

If the characteristic equation $x^2 - r_1x - r_2$ of the reference relation $a_n = r_1 a_{n-1} + r_2 a_{n-2}$ has two distinct roots s_1 and s_2. Then $a_n = us_1^n + vs_2^n$ where u and v are initial conditions is the explicit formula for the sequence.

(b) If the characteristic equation $x^2 - r_1x - r_2$ has a single root s then explicit formula is $a_n = us^n + vns^n$. where u and v are initial condition

Proof :

If s_1 and s_2 are two roots of ch. equn.

$x^2 - r_1x - r_2 = 0$.

$\therefore s_1^2 - r_1s_1 - r_2 = 0 \rightarrow (1)$

$s_2^2 - r_1s_2 - r_2 = 0 \rightarrow (2)$.

$(1) \Rightarrow \quad s_1^2 = r_1s_1 + r_2 \rightarrow (3)$

$(2) \Rightarrow \quad s_2^2 = r_1s_2 + r_2 \rightarrow (4)$

Given explicit formula is $a_n = us_1^n + vs_2^n$

$\therefore a_n = u(s_1^{n-2})(s_1^2) + v(s_2^{n-2})(s_2^2)$

sub (3) + (4) $\Rightarrow a_n = u(s_1^{n-2})(r_1s_1 + r_2) + v(s_2^{n-2})(r_1s_2 + r_2)$

$\therefore a_n = ur_1s_1^{n-1} + ur_2s_1^{n-2} + vr_1s_2^{n-1} + vr_2s_2^{n-2}$

$= r_1[us_1^{n-1} + vs_2^{n-1}] + r_2[us_1^{n-2} + vs_2^{n-2}]$

$a_n = r_1a_{n-1} + r_2a_{n-2}$ is a required recurrence relation.

(b) If S is the root of $x^2 - r_1 x - r_2 = 0$.

$$S^2 - r_1 S - r_2 = 0.$$

$$S^2 = r_1 S + r_2 \rightarrow (1).$$

$\therefore$ Explicit formula is $a_n = uS^n + vnS^n$.

$$= u(S^{n-2})(S^2) + vn(S^{n-2})(S^2)$$

$$= u(r_1 S + r_2)(S^{n-2}) + vn(r_1 S + r_2)(S^{n-2})$$

$$= ur_1 S^{n-1} + ur_2 S^{n-2} + vn r_1 S^{n-1} + vn r_2 S^{n-2}$$

$$= r_1[uS^{n-1} + vn r_1 S^{n-1}] + r_2[uS^{n-2} + vnS^{n-2}]$$

$$= r_1[a_{n-1}] + r_2[a_{n-2}]$$

$a_n = r_1 a_{n-1} + r_2 a_{n-2}$ is the recurrence relation.

1/2/2020.

Find the explicit formula for the sequence defined by $C_n = 3C_n - 2C_{n-2}$ with initial condition $C_1 = 5, C_2 = 3$.

Sol: Given recurrence relation is

$$C_n = 3C_n - 2C_{n-2}$$

$$\therefore r_1 = 3,\ r_2 = -2$$

Characteristic equation is $x^2 - r_1 x - r_2 = 0$.

$$\Rightarrow x^2 - 3x + 2 = 0.$$

$$(x-1)(x-2) = 0.$$

$$\therefore x_1 = 1, 2.$$

The distinct roots are $S_1 = 1,\ S_2 = 2$.

The explicit formula is

$$C_n = uS_1^n + vS_2^n$$

$n=1$

$$C_1 = uS_1 + vS_2$$

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$\Rightarrow 5 = u(1) + v(2)$

$u + 2v = 5 \rightarrow (1)$

Now $n = 2$,

$c_2 = u s_1^2 + v s_2^2$

$3 = u(1)^2 + v(2)^2$

$u + 4v = 3 \rightarrow (2)$

To find u & v, by solving (1) & (2).

$u + 2v = 5$

$u + 4v = 3$

$-2v = 2$

$\boxed{v = -1}$ in (1)

$u - 2 = 5$

$\boxed{u = 7}$

$\therefore$ Required explicit formula is

$c_n = 7(1)^n - (2)^n$

$c_n = 7 - 2^n$ $\boxed{c_n = 7 - 2^n}$

The Required sequence is $5, 3, -1, \ldots$

2. Find explicit formula for $d_n = 2d_{n-1} - d_{n-2}$ with initial condition $d_1 = 1.5$ & $d_2 = 3$.

Solution:

Given recurrence equation is $d_n = 2d_{n-1} - d_{n-2}$

$\therefore r_1 = 2$ $r_2 = -1$

Characteristic equation is $x^2 - r_1 x - r_2 = 0$

$\Rightarrow x^2 - 2x + 1 = 0$.

$\Rightarrow (x-1)(x-1) = 0$

$\Rightarrow x = 1, 1$

The two roots are $s_1 = 1$, $s_2 = 1$

The explicit formula is $a_n = u s_1^n + v n s_2^n$

$n = 1$,

$a_1 = u(1)^1 + v(1)^1 \Rightarrow 1.5 = u + v \rightarrow (1)$.

$n = 2$,

$a_2 = u(1)^2 + 2v(1)^2 \Rightarrow 3 = u + 2v \rightarrow (2)$.

To find u & v by solving (1) & (2).

$u + v = 1.5$

$u + 2v = 3$

$-v = -1.5$

$\boxed{v = 1.5}$

Sub $v = 1.5$ in (1)

$1.5 + u = 1.5$

$\boxed{u = 0}$

The required explicit formula is

$a_n = (1.5) n (1)^n$

$= (1.5) n$.

$\therefore$ The sequence is $1.5, 3, 4, 5, \ldots$

Solve by backtrack $a_n = 2.5\, a_{n-1}$ initial Condition

$a_1 = 4$

Find the explicit formula for the sequence eqn ~~Expression of the~~

$a_n = 4a_{n-1} + 5a_{n-2}$

$a_1 = 2$, $a_2 = 6$

Given recurrence equation is $a_n = 4a_{n-1} + 5a_{n-2}$

$\therefore r_1 = 4 \quad r_2 = 5$

Ch. eqn. is $x^2 - r_1 x - r_2 = 0$

$\Rightarrow x^2 - 4x - 5 = 0$.

$\Rightarrow (x-5)(x+1) = 0$

$x = 5, -1$

The two roots are $s_1 = -1$, $s_2 = 5$.

The explicit formula is $a_n = u s_1^n + v s_2^n$.

$n = 1$

$a_1 = u s_1^1 + v s_2^1 \Rightarrow 2 = u + 5v \rightarrow (1)$

$n = 2$

$a_2 = u s_1^2 + v s_2^2 \Rightarrow 6 = u + 25v \rightarrow (2)$.

To find u and v by solving (1) & (2)

$$\begin{aligned} -u + 5v &= 2 \\ u + 25v &= 6 \\ \hline +30v &= +8 \end{aligned}$$

$\boxed{v = +\frac{4}{15}}$ in (1)

$-u + 5\left(+\frac{4}{15}\right) = 2$

$-u + \frac{4}{3} = 2$

$-u + 5\left(\frac{4}{15}\right) = 2 \Rightarrow$

$\boxed{u = -\frac{2}{3}}$

The required explicit formula is

$$a_n = -\frac{2}{3}(-1)^n + \frac{4}{15}(5)^n.$$

), $a_n = 2.5\, a_{n-1}$, $a_1 = 4$.

$\Rightarrow a_n = 2.5\, a_{n-1}$

$= 2.5\, [2.5\, a_{n-2}]$

$= 2.5^2\, a_{n-2}$

$= (2.5)^2 \cdot [2.5\, a_{n-3}]$

$= 2.5^3\, a_{n-3}$ and so on.

$\therefore\ a_n = (2.5)^{n-1}\, a_{n-(n-1)}$

$= (2.5)^{n-1}\, a_1$

$\boxed{\therefore a_n = 4\,(2.5)^{n-1}}$

$\therefore$ Required sequence $a_1 = 4, 10, \ldots$

relation

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5.2.2020

UNIT - II

RELATIONS AND DIGRAPHS:

The theory of relation was given by 'Augustus De Morgan'.

Digraps was given by British philosopher and mathematician "Bertrand Russell" in 1919

4.1 Product sets and Partitions:

The Cartesian product of $A = \{1,2,3\}$, $B = \{a,b\}$.

$$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$$

The general formula:

$$i.e., \; A \times B = \{(a,b) \mid a \in A, b \in B\}.$$

Example: Find $A \times A$ if $A = \{1,2,3\}$ $B = \{r,s\}$

solution:

$$B \times A = \{(r,1), (r,2), (r,3), (s,1), (s,2), (s,3)\}.$$

Partitions: (Quotient set)

A partition or Quotient set of a non-empty set A is a collection P of non-subsets of A such that,

i) each element of A belongs to one of the sets in P

ii) If A_1 and A_2 are distinct elements of P then $A_1 \cap A_2 = \{\phi\}$.

iii) The sets in P are called blocks or cells of the partition

Example: Let $A = \{a,b,c,d,e,f,g,h\}$. Consider the subsets of A are $A_1 = \{a,b,c,d\}$, $A_2 = \{a,c,e,f,g,h\}$, $A_3 = \{a,c,e,g\}$, $A_4 = \{b,d\}$, $A_5 = \{f,h\}$.

Solution

$\{A_2, A_4\}$ are partition of A

because (i) $A_2 \cap A_4 = \phi$.

& (ii) $A_2 \cup A_4 = A$

and $\{A_3, A_4, A_5\}$ is a partition of A

because (i) $A_3 \cap A_4 = \phi$, $A_4 \cap A_5 = \phi$, $A_3 \cap A_5 = \phi$

& (ii) $A_3 \cup A_4 \cup A_5 = A$.

Example:

Let Z = {set of all integers

A_1 = set of all even numbers

A_2 = set of all odd numbers.

Solution:

$\{A_1, A_2\}$ are partition of Z

because (i) $A_1 \cap A_2 = \phi$.

(ii) $A_1 \cup A_2 = Z$.

4.2 Relation & Digraphs

Let A and B be non empty set a relation R from A to B is a subset of $A \times B$.

If $R \subseteq A \times B$ & $(a, b) \in R$

Then a is related to b.

i.e, $a R b$

Example:

1 Let $A = \{1, 2, 3\}$ & $B = \{r, s\}$. Then $R = \{(1, r), (2, s), (3, r)\}$ is a relation from A to B

Let A and B be set of real numbers we defined the relation R from A to B aRb iff $a = b$.

Solution

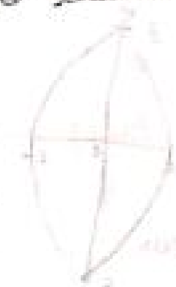
$R = \{(a, b) \mid a = b$ of $a \in A, b \in B\}$

2 Let $A = \{1, 2, 3, 4, 5\}$ Define the relation R ~~of~~ by aRb iff $a < b$.

Solution: $R = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$.

$Dom(R) = \{1, 2, 3, 4\}$ & $Ran(R) = \{2, 3, 4, 5\}$.

3 Let $A = \mathbb{R}$ the set of real numbers defined relation R on A by xRy iff $\frac{x^2}{4} + \frac{y^2}{9} = 1$.



Solution: $R = \{(0,3), (0,-3), (2,0), (-2,0)\}$.

Definition: (a) $\longrightarrow$ (b).

The domain of ~~R~~ R denoted by $Dom(R)$ is the set of elements in A.

The range of R is denoted by $Ran(R)$ is the set of elements in B.

If R is the relation defined by $R\{(a,b), (c,d), (e,f)\}$. Then $Dom\, R = \{a, c, e\}$.
$Ran\, R = \{b, d, f\}$.

4. If R is the relation defined by $R = \{(1,8), (2,5), (3,7)\}$ where $A = \{1, 2, 3\}$ and $B = \{7, 8\}$. Find Domain & Range.

Solution:

$Dom(R) = \{1, 2, 3\} = A$

$Ran(R) = \{7, 9\} = B$

5. If $A = \{a, b, c, d\}$. $R = \{(a,a), (a,b), (b,c), (c,a), (d,c), (c,b)\}$. Then find $R(a)$, & $R(b)$ if $A_1 = \{c, d\}$. Find $R(A_1)$.

Solution: $R(a) = \{a, b\}$, $R(b) = \{c\}$.

$R\{A_1\} = \{a, b, c\}$.

Theorem:

Let R be a relation from A to B and A_1 and A_2 be subsets of A then (a) If $A_1 \subseteq A_2$ Then $R(A_1) \subseteq R(A_2)$. (b) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$ (c) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$.

Proof:

(a) If $A_1 \subseteq A_2$

If $y \in R(A_1)$

Then $\exists$ some $x \in A_1$, s.t xRy.

But $A_1 \subseteq A_2 \Rightarrow x \in A_2$.

Also $xRy \Rightarrow y \in R(A_2)$

$\therefore R(A_1) \subseteq R(A_2)$

(b) If $y \in R(A_1 \cup A_2)$

$\exists$ $x \in A_1 \cup A_2$ s.t xRy.

$\Rightarrow x = A_1$ (or) $x \in A_2$

If $x \in A_1$ & xRy.

$\Rightarrow y \in R(A_1)$

If $x \in A_2$ & xRy

$\Rightarrow y \in R(A_2)$

$y \in R(A_1)$ (or) $y \in R(A_2)$

$\Rightarrow y \in R(A_1) \cup R(A_2)$

$\therefore R(A_1 \cup A_2) \subseteq R(A_1) \cup R(A_2) \rightarrow$ ①

If $y \in R(A_1) \cup R(A_2)$

$\Rightarrow y \in R(A_1)$ (or) $y \in R(A_2)$

If $y \in R(A_1)$

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$\exists$ some $x \in A_1$ s.t xRy.

If $y \in R(A_2)$

$\exists\ x \in A_2$ s.t xRy

$\therefore x \in A_1$ (or) $x \in A_2$ with xRy.

$\Rightarrow x \in A_1 \cup A_2$ s.t xRy.

$\Rightarrow y \in R(A_1 \cup A_2)$

$\therefore R(A_1) \cup R(A_2) \subseteq R(A_1 \cup A_2)$

from ① + ⑤. $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(c) If $y \in R(A_1 \cap A_2)$

$\Rightarrow \exists\ x \in A_1 \cap A_2$ s.t xRy.

$\Rightarrow x \in A_1$ & $x \in A_2$ s.t xRy

$\Rightarrow x \in A_1$ with xRy.

& $x \in A_2$ with xRy

$\therefore y \in R(A_1)$ and $y \in R(A_2)$.

$\Rightarrow y \in R(A_1) \cap R(A_2)$

$\therefore R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$ //.

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Example:

1. Let $A = \{1,2,3\}$ $B = \{x, y, z, w, p, q\}$ Relation $R = \{(1,x), (1,z), (2,w), (2,p), (2,q), (3,y)\}$. Let $A_1 = \{1,2\}$ and $A_2 = \{2,3\}$. P T $R(A_1 \cap A_2) = R(A_1) \cap R(A_2)$.

$A_1 \cap A_2 = \{2\}$

$R(A_1 \cap A_2) = \{w, p, q\} \rightarrow ①$

$R(A_1) = \{x, z, w, p, q\}$

$R(A_2) = \{w, p, q, y\}$.

$R(A_1) \cap R(A_2) = \{w, p, q\} \rightarrow ②$

From ① and ②

$R(A_1 \cap A_2) = R(A_1) \cap R(A_2)$

(2) If R and S are relations from A to B. If $R(a) = S(a)$ $\forall a \in A$ then $R = S$.

Proof:

If aRb when $a \in A$

$b \in R(a)$

Since $R(a) = S(a)$

$\therefore b \in S(a)$ & aSb

$\therefore R(a) \subseteq S(a)$

lly, we can prove

$S(a) \subseteq R(a)$

$\therefore R(a) = S(a)$ $\forall a \in A$.

$\therefore R = S$.

Matrix of a Relation.

If $A = \{a_1, a_2, a_3, \dots a_m\}$; $B = \{b_1, b_2, \dots b_n\}$ are finite sets and R is a relation from A to B then M_R is denoted by $[m_{ij}]$. Here $M_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$

Example: Let $A = \{1, 2, 3\}$ $B = \{7, 5\}$

$R = \{(1,7), (2,5), (3,7)\}$ Find M_R.

Solution:

$$M_R = \begin{array}{c c} & \begin{array}{cc} 7 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

Consider a matrix $M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ Find a relation.

Solution :

$$
\begin{array}{cc}
 & \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\
M = \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

Let $A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3, b_4\}$.

$\therefore$ Relation $R = \{(a_1, b_1), (a_1, b_4)$

$(a_2, b_2), (a_2, b_3)$

$(a_3, b_1), (a_3, b_3)\}$.

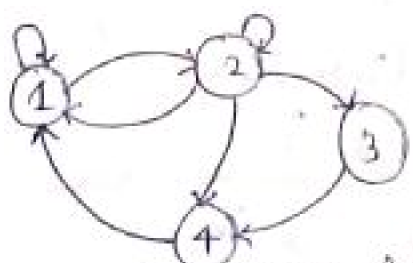
Diagraph of a Relation :

Let A is a finite set R is relation on A we can represent R pictorially as draw a small circle for each element of A and these circles are called vertices and draw an arrow called edge from a vertex a_i to a vertex a_j iff $a_i R a_j$.

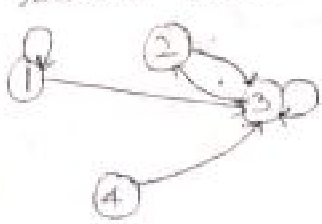
Example :

Let $A = \{1, 2, 3, 4\}$ $R = \{(1,1), (1,2), (2,1), (2,2),$ $(2,3), (2,4), (3,4), (4,1)\}$ find a diagraph of R.

Solution :



Find the relation determined from the diagram



Solution :

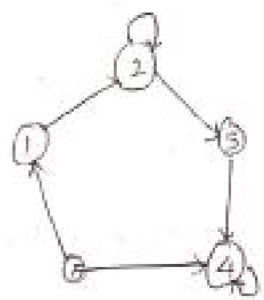
$R = \{(1,1), (1,3), (2,3), (3,2), (3,3), (4,3)\}$

2. Let $A = \{1, 2, 3, 4\}$ $B = \{1, 4, 6, 8, 9\}$ $A R B$ iff $B = A^2$. find the relation.

3. Let $A = \{1, 2, 3, 4\}$ and $M_R = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ find R.

and diagraph of R.

4. From this

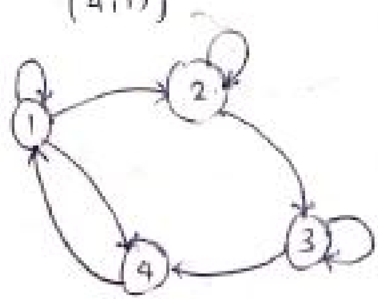


find R and M_R

2. $R = \{(2,4), (1,1), (3,9)\}$

3.
$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

$R = \{(1,1), (1,2), (1,4),$
$(2,2), (2,3),$
$(3,3), (3,4)$
$(4,1)\}$



4. $R = \{(1,2), (2,2), (2,3), (3,4), (4,4), (5,4), (5,1)\}$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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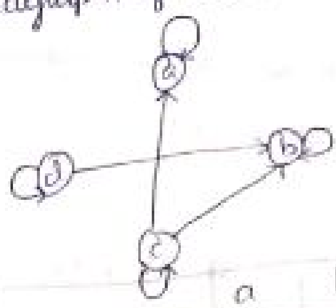
1. Let $A = \{a, b, c, d\}$ & let R be the relation on A has the metric

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Construct a diggraph of R and list indegree and out degree.

Solution :

Diagraph of R is



	a	b	c	d
In degree	2	3	1	1
out degree	1	1	3	2

Eg :

Let $A = \{1, 4, 5\}$ and R be a graph as

find M_R & R

$$M_R = \begin{array}{c} \\ 1 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

$R = \{(1,4), (1,5), (4,1), (4,4), (5,4), (5,5)\}$.

Define:

If R is a relation on a set A and B is a subset of A the restriction of R to B is $R \cap (B \times B)$.

Example:

Let $A = \{a, b, c, d, e, f\}$ $R = \{(a,a), (a,c), (b,c), (a,e), (b,e), (c,e)\}$ $B = \{a, b, c\}$. Then find restriction of R to B.

Solution:

$$B \times B = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$$

$$R = \{(a,a), (a,c), (b,c), (a,e), (b,e), (c,e)\}$$

$$\therefore R \cap (B \times B) = \{(a,a), (a,c), (b,c)\}$$

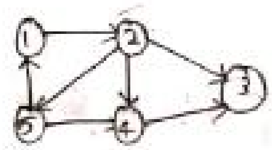
4.3. paths in relation & diagraph.

path:

Suppose that R is a relation on A a path of length n in R from a to b is a finite sequence π from $a, x_1, x_2 \ldots x_{n-1}, b$. Beginning with a and ending with b. s.t $a R x_1, x_1 R x_2 \ldots x_{n-1} R b$

Example:

Consider the diagraph.



Solution:

π_1 : 1, 2, 5, 4, 3 is a path of length 4.

π_2 : 1, 2, 5 & 1, 2, 4, 3 are a path of length

π_3 : 1, 2, 3 4 5, 4, 3 are path of length 2.

Theorem:

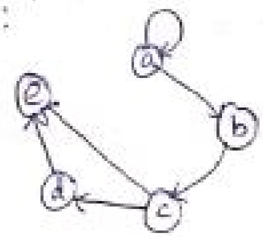
If R is a relation on $A = \{a_1, a_2, \dots a_n\}$

Then $M_{R^2} = M_R \odot M_R$.

Example:

Let $A = \{a, b, c, d, e\}$ and $R = \{(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)\}$ Compute (a) R^2 (b) R^∞.

Solution:



aR^2b are aRa & aRb.

bR^2d are bRc & cRd.

aR^2c are aRb & bRc.

bR^2e are bRc & cRe.

cR^2e are cRd & dRe.

(a) : $R^2 = \{(a,b), (b,d), (a,c), (b,e), (c,e)\}$.

(b) $R^\infty = \{(a,a), (a,b), (a,c), (a,d), (a,e), (b,e), (b,d), (b,c), (c,d), (c,e), (d,e)\}$.

Example:

Let $A = \{a, b, c, d, e\}$

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$R = \{(a,a), (a,b), (b,c), (c,e), (c,d), (d,e)\}$.

Find M_{R^2}.

Solution:

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$M_{R^2} = M_R \odot M_R$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{R^2} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4.4. Properties of Relation:

Reflexive & irreflexive relation

A relation R on a set A is reflexive if [(a, a) is in R] $(a, a) \in R \ \forall a \in A$.

A relation R on a set A is irreflexive if $a \not R a \ \forall a \in A$.

Example:

(1) Let $R = \{(a, a) \mid a \in A\}$ where R is a relation defined as $a = b$. Then R is ~~refractive~~ reflexive

Yes, R is reflexive

(2) Let $A = \{1, 2, 3\}$ & $R = \{(1, 1), (1, 2)\}$.

Then R is irreflexive.

↳ $R = \{(1,1), (2,2), (3,3)\}$ it is reflexive

Symmetric :

A relation R on a set A is symmetric if whenever a R b then b R a.

Asymmetric :

A relation R on a set A is Asymmetric if whenever a R b then b $\not R$ a.

Antisymmetric :

A relation R on a set A is Antisymmetric if whenever a R b and b R a then a = b

Eg:

Let A = $\mathbb{Z}$, the set of all integers. Let R = $\{(a,b) \in A \times A / a < b\}$

Is R symmetric, asymmetric & antisymmetric

solution:

(a) R is not symmetric because $4 < 5$, but $5 \not< 4$

(b) R is Asymmetric because, $2 < 3$ ($2 R 3$), but $3 \not< 2$ ($3 \not R 2$).

(c) R is ~~also~~ not Antisymmetric

Example:

Let A = {1, 2, 3, 4}. Let R = {(1,2), (2,2), (3,4), (4,1)}

Is R symmetric, Asymmetric, Antisymmetric.

Solution:

(a) R is not symmetric because $(1,2) \in R$ but $(2,1) \notin R$.

(b) R is Asymmetric because $(1,2) \in R$ & $(2,1) \notin R$.

(c) R is antisymmetric because $(1,3) \notin R$ & $(3,1) \notin R$ then $3 \neq 1$. i.e., $(a,b) \notin R$

$(b,a) \notin R$

Then $a \neq b$.

Example:

(i) $M_{R_1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ (ii) $M_{R_2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ (iii) $M_{R_3} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(iv) $M_{R_4} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ (v) $M_{R_5} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ (vi) $M_{R_6} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Check this for reflexive, irreflexive, symmetric, Asymmetric and Antisymmetric.

(i) $M_{R_1} = \{(a,a), (a,c), (b,c), (c,a), (c,b), (c,c)\}$

R_1 is irreflexive, symmetric, Antisymmetric

(ii) $M_{R_2} = \{(a,b), (a,c), (b,a), (b,b), (c,a), (c,c), (c,d), (d,c), (d,d)\}$.

R_2 is irreflexive, symmetric, Antisymmetric

(iii) $M_{R_3} = \{(a,a), (a,b), (a,c), (b,b)\}$.

R_3 is irreflexive, Asymmetric,

(iv) $M_{R_4} = [(a,b), (a,c), (b,a), (b,b), (c,a), (c,c), (c,d), (d,c), (d,d)\}$.

R_4 is irreflexive, symmetric, Antisymmetric.

(v) $M_{R_5} = \{(a,c), (a,d), (b,c), (c,d), (d,a)\}$.

R_5 is irreflexive, symmetric, Antisymmetric

(vi) $M_{R_6} = \{(a,b), (a,c), (a,d), (b,c), (c,d)\}$.

R_6 is irreflexive, Asymmetric

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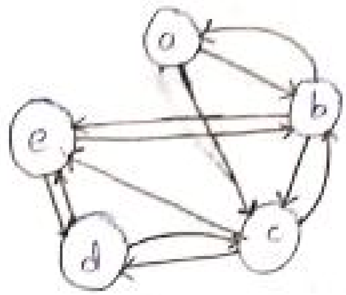
Define:

To vertices a and b are connected by edge in both directions is called graph of the symmetric relation.

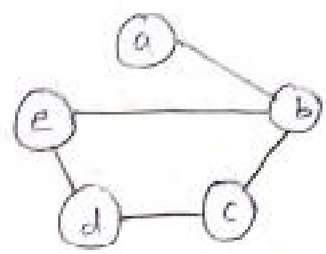
Eg: let $A = \{a,b,c,d,e\}$ and $R = \{(a,b),(b,a),(a,c),(c,e),$ $(b,c),(c,b),(b,e),(e,b),(e,d),(d,e),(c,d),(d,c)\}$.

Find the graph of R.

Sol: digraph of R is



The graph of R is



Transitive relation:

A relation R on a set A is said to be transitive if whenever aRb, bRc. Then aRc.

Example:

Let $A = \mathbb{Z}$, the set of all integers, R be a relation "less than", find whether R is transitive.

Solution:

Clearly $a<b$, $b<c$. Then $a<c$.

for all $a,b,c \in \mathbb{Z}$

$\therefore$ R is transitive

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2. Let $A = \{1,2,3,4\}$ $R = \{(1,2), (1,3), (4,2)\}$ Is R transitive.

Solution:

R is not transitive because $(1,4) \notin R$ & $(4,3) \notin R$ but $(1,3) \in R$.

3. Let $A = \{1,2,3\}$ $M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ ST R is transitive.

$$R = \{(1,1), (1,2), (1,3), (2,3), (3,3)\}.$$

$$\left.\begin{array}{l} (1,1) \in R \\ \& \ (1,2) \in R \end{array}\right\} (1,2) \in R.$$

$$\left.\begin{array}{l} (1,2) \in R \\ (2,3) \in R \end{array}\right\} \Rightarrow (1,3) \in R.$$

$$\left.\begin{array}{l} (2,3) \in R \\ (3,3) \in R \end{array}\right\} \Rightarrow (2,3) \in R.$$

$$\left.\begin{array}{l} (1,3) \in R \\ (3,3) \in R \end{array}\right\} \Rightarrow (1,3) \in R.$$

Second Method:

If $M_R^2 = M_R$. Then R is transitive

Example:

$$M_R^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = M_R.$$

4.5 Equivalence relation:

A relation R on a set A is called the equivalence relation. If it is reflexive, symmetric and transitive

Example:

Let $A = \{1, 2, 3, 4\}$. and $R = \{(1,1), (1,2), (2,1), (2,2)$ $(3,4), (4,3), (3,3), (4,4)\}$. check whether R is a equivalence relation.

Sol: R is reflexive [(1,1),(2,2),(3,3),(4,4)], symmetric & transitive.

$\therefore$ R is the equivalence relation

THEOREM:

Let P be a partition of a set A, define the relation R on A as follows. aRb iff a and b are in same block. Then R is an equivalence relation

Sol:

(i) If $a \in A$, then clearly a is a in same block.

$\therefore aRa$.

$\therefore$ R is reflexive.

(ii) If aRb, Then a & b are in same block

b & a also in same block.

$\therefore bRa$. $\therefore$ R is symmetric

(iii) If aRb & bRc.

Then a & b are in same block.

b & c are in same block.

$\therefore$ a & c are in same block.

$\therefore aRc$.

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.

Eg: Let $A = \{1,2,3,4\}$ consider partition $P = \{\{1,2,3\},\{4\}\}$ of A. find the equivalence relation R on A determined by P.

A.T

Sol:

$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(1,3),(2,3),(3,1),(1,4)\}$ ...

Theorem :

Let R be an equivalence relation on a set A, $a \in A$, & $b \in A$. Then aRb iff $R(a) = R(b)$.

Proof :

If $R(a) = R(b)$.

$a \in R(a)$.

$a \in R(b)$

Then, aRb.

Conversely, if $aRb \Rightarrow b \in R(a)$.

Since R is symmetric

$bRa \Rightarrow a \in R(b)$

Also R is reflexive

$a \in R(a)$

$R(b) \subseteq R(a) \rightarrow (1)$.

Suppose $a \in R(a)$

Given $aRb \Rightarrow bRa$.

$\Rightarrow a \in R(b)$

$\therefore R(a) \subseteq R(b) \rightarrow (2)$

$\therefore R(a) = R(b)$.

of A.

4.7 Operations on relations:

Eg: Let $A = \{1, 2, 3, 4\}$ & $B = \{a, b, c\}$.

Let $R = \{(1,a), (1,b), (2,b), (2,c), (3,b), (4,a)\}$.

& $S = \{(1,b), (2,c), (3,b), (4,b)\}$. Compute

(a) $\overline{R}$ (b) $R \cap S$ (c) $R \cup S$ (d) R^{-1}.

<u>Solution:</u>

~~$A \times A = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4)$
$(2,1), (2,3), (2,4), (3,1), (3,2), (3,4)$
$(4,1), (4,2), (4,3)\}$.~~

(a) $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b),$
$(2,c), (3,a), (3,b), (3,c), (4,a),$
$(4,b), (4,c)\}$.

$\overline{R} = \{(1,c), (2,a), (3,a), (3,c), (4,b), (4,c)\}$.

(b) $R \cap S = \{(1,b), (2,c), (3,b)\}$.

(c) $R \cup S = \{(1,a), (1,b), (2,b), (2,c), (3,b), (4,a), (4,b)\}$.

(d) $R^{-1} = \{(a,1), (b,1), (2,b), (c,2), (b,3), (a,4)\}$.

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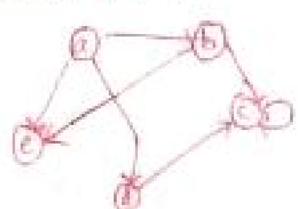
Eg: Let $A = \{a, b, c, d, e\}$. Let R & S be the two relations on A corresponding to the digraphs.



R

S

find $\overline{R}$, R^{-1} & $R \cap S$

$R\{(a,b), (b,c), (c,c), (b,d), (d,c), (a,d), (a,e),$
$(b,e), (c,e), (d,d)\}$

$S = \{(a,a), (a,b), (b,b), (b,e), (c,c), (c,b), (c,a),$
$(d,b), (e,a), (e,d)\}.$

$A \times A = \{(a,a), (a,b), (a,c), (a,d), (a,e),$
$(b,a), (b,b), (b,c), (b,d), (b,e),$
$(c,a), (c,b), (c,c), (c,d), (c,e),$
$(d,a), (d,b), (d,c), (d,d), (d,e),$
$(e,a), (e,b), (e,c), (e,d), (e,e)\}.$

$\bar{R} = \{(a,a), (a,c), (b,a), (b,b), (c,a), (c,b),$
$(c,d), (c,e), (d,a), (d,b), (d,e), (e,b), (e,c)$
$(e,a), (e,d)\}$

$R^{-1} = \{(b,a), (c,b), (c,c), (d,b), (c,d), (d,a), (e,a)$
$(e,b), (e,c), (d,d)\}.$

$R \cap S = \{(a,b), (c,c), (b,e)\}.$

Eg: Let $A = \{1,2,3\}$ & R & S are relations on A

The matrices of R & S are $M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $M_S = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

find $M_{\bar{R}}$, $M_{R^{-1}}$, $M_{R\cap S}$ & $M_{R\cup S}$

Solution:

$$M_{\bar{R}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad M_{R^{-1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$M_{R\cap S} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_{R\cup S} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Theorem

Suppose that R and S are relations from A to B

(i) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$

(ii) If $R \subseteq S$, then $\bar{S} \subseteq \bar{R}$.

(iii) $(R\cap S)^{-1} = R^{-1} \cap S^{-1}$ & $(R\cup S)^{-1} = R^{-1} \cup S^{-1}$

(iv) $(\overline{R\cap S}) = \bar{R} \cup \bar{S}$ & $\overline{R\cup S} = \bar{R}\cap\bar{S}$

Sol:

(i) If $R \subseteq S$

If $(a,b) \in R^{-1}$

$\Rightarrow (b,a) \in R$

$(b,a) \in S$

$(a,b) \in S^{-1}$

$\therefore R^{-1} \subseteq S^{-1}$

(ii) $R \subseteq S$

If $(a,b) \in \bar{S}$

~~b~~ $(a,b) \notin S$

$(a,b) \notin R$

$(a,b) \in \bar{R}$

$\therefore \bar{S} \subseteq \bar{R}$

(iii) If $(a,b) \in (R\cap S)^{-1}$

$\Rightarrow (b,a) \in R\cap S$

$(b,a) \in R$ & $(b,a) \in S$

$(a,b) \in R^{-1}$ & $(a,b) \in S^{-1}$

$\Rightarrow (a,b) \in R^{-1} \cap S^{-1}$

$(R\cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$

$||^{ly}$ we prove

$(R\cup S)^{-1} = R^{-1} \cup S^{-1}$

(iv) ~~If $(a,b) \in (\overline{R\cap S})$~~

~~$\Rightarrow (b,a) \notin (R\cap S)$~~

If $(a,b) \in (R\cup S)^{-1}$

$(b,a) \in R\cup S$

$(b,a) \in R$ & $(b,a) \in S$

$(a,b) \in R^{-1}$ & $(a,b) \in S^{-1}$

$(R\cup S)^{-1} \subseteq R^{-1} \cup S^{-1}$

$\therefore (R\cup S)^{-1} = R^{-1} \cup S^{-1}$

(iv) $(a,b) \in (\overline{R \cap S})$

$(a,b) \notin R \cap S$

$(a,b) \notin R$ and $(a,b) \notin S$

$(a,b) \in \overline{R}$ or $(a,b) \in \overline{S}$

$(a,b) \in \overline{R} \cup \overline{S}$

$(\overline{R \cap S}) \subseteq \overline{R} \cup \overline{S}$

$\overline{R \cap S} = \overline{R} \cup \overline{S}$

Ily

$(a,b) \in (\overline{R \cup S})$

$(a,b) \notin (R \cup S)$

$(a,b) \notin R$ and $(a,b) \notin S$

$(a,b) \in \overline{R}$ and $(a,b) \in \overline{S}$

$(a,b) \in \overline{R} \cap \overline{S}$

$\overline{R \cup S} \subseteq \overline{R} \cap \overline{S}$

$\overline{R \cup S} = \overline{R} \cap \overline{S}$.

25/2/2020 Theorem :

Let R & S be relations on a set A

(a) If R is reflexive, so is R^{-1}

(b) If R & S are reflexive, then $R \cap S$ & $R \cup S$ are also reflexive

(c) R is reflexive iff $\overline{R}$ is irreflexive.

Proof :

(a) If $(a,a) \in R \quad \forall a \in R$.

$\Rightarrow (a,a) \in R^{-1}$

$\therefore R^{-1}$ is reflexive

(b) If R and S are reflexive

Then $(a,a) \in R$ & $(a,a) \in S \ \forall a \in R$.

$(a,a) \in R \cup S$

& $(a,a) \in R \cap S$.

$\therefore R \cup S$ & $R \cap S$. are reflexive .

(c) If $(a,a) \in R$.

$(a,a) \notin \bar{R}$, $\forall a \in A$.

$\therefore \bar{R}$ is irreflexive.

Conversely, If $\bar{R}$ is irreflexive .

then, $(a,a) \notin \bar{R}$

$(a,a) \in R$.

$\therefore R$ is reflexive .

eg: Let $A = \{1,2,3\}$ & consider two reflexive relation . $R = \{(1,1),(1,2),(1,3),(2,2),(3,3)\}$.

& $S = \{(1,1),(1,2),(2,2),(3,2),(3,3)\}$

Find R^{-1}, $R \cup S$ & $R \cap S$ are also reflexive & $\bar{R}$ is Irreflexive .

Proof : $R^{-1} = \{(1,1),(2,1),(3,1),(2,2),(3,3)\}$.

R^{-1} is reflexive.

$R \cup S = \{(1,1),(1,2),(2,2),(3,3),(3,2)\}$ is reflexive

$R \cap S = \{(1,1),(2,2),(3,3)\ (1,2)\}$ is reflexive

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1)$
$(3,2), (3,3)\}$.

$\bar{R} = \{(1,3), (2,1), (2,3), (3,1), (3,2)\}$.

$\bar{R}$ is irreflexive,

Theorem:

Let R be a relation on Set in A, then (a) R is makoue symmetric iff $R = R^{-1}$

(b) R is anti symmetric iff $R \cap R^{-1} \subseteq S$

(c) R is asymmetric iff $R \cap R^{-1} = \emptyset$.

(a) R is symmetric iff $R = R^{-1}$

If R is symmetric

$(a,b) \in R$.

$\Leftrightarrow (b,a) \in R$.

$\Leftrightarrow (b,a) \in R^{-1}$

$R = R^{-1}$

Conversely, if $R = R^{-1}$

If $(a,b) \in R$.

$\Rightarrow (b,a) \in R^{-1} = R \quad (\because R = R^{-1})$

$(b,a) \in R$

$\therefore$ R is symmetric

(b) R is Antisymmetric iff $R \cap R^{-1} \subseteq \Delta$.

Proof: R is Antisymmetric

$(a,b) \in R$ & $(b,a) \in R$.

$\Rightarrow a = b$

$\Rightarrow (a,a) \in R$.

Scanned with CamScanner

$\therefore (a,a) \in R^{-1}$

$\therefore (a,a) \in R \cap R^{-1}$

$\therefore R \cap R^{-1} \subseteq \Delta$

Conversely, if $R \cap R^{-1} \subseteq \Delta$.

$\Rightarrow (a,a) \in R$ & $(a,a) \in R^{-1}$

$\Rightarrow a = b$.

$\therefore R$ is anti symmetric

(C) R is Asymmetric iff $R \cap R^{-1} = \phi$.

Sol: If R is Asymmetric

$\Rightarrow (aRb)$ but $b \not R a$.

i.e, $(a,b) \in R$.

$(b,a) \notin R$.

But $(b,a) \in R^{-1}$

$\therefore R \cap R^{-1} = \phi$.

Conversely, if $R \cap R^{-1} = \phi$.

$\Rightarrow (a,b) \in R$.

$(b,a) \in R^{-1}$

But $R \cap R^{-1} = \phi \Rightarrow aRb$ but $b \not R a$.

$\therefore R$ is Asymmetric.

Theorem:

If R & S be relations on A.

(a) If R is symmetric so is R^{-1} & $\overline{R}$

(b) If R & S are symmetric, so is $R \cap S$ & $R \cup S$

Sol:
(a) If R is symmetric.

$R = R^{-1}$

$\Rightarrow (R^{-1})^{-1} = R^{-1}$

$\Rightarrow R^{-1} = (R^{-1})^{-1}$

R^{-1} is symmetric

If $(a,b) \in R \Rightarrow (b,a) \in R$.

$\Rightarrow (a,b) \notin \bar{R} \Rightarrow (b,a) \notin \bar{R}$.

$\therefore \bar{R}$ is symmetric

(b) If R & S are symmetric.

If $(a,b) \in R$.

$(b,a) \in R$.

& $(a,b) \in S$

$(b,a) \in S$.

$R \cup S = \{(a,b), (b,a)\}$. $\therefore R \cup S$ is symmetric

$R \cap S = \{(a,b), (b,a)\}$

$\therefore R \cap S$ is symmetric

Example: Let $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,1), (1,3), (3,1)\}$

& $S = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

Then prove R^{-1}, S^{-1} & $R \cup S$ and $R \cap S$ are symmetric

$R^{-1} = \{(1,1), (2,1), (1,2), (3,1), (1,3)\}$.

R^{-1} is Symmetric

$S^{-1} = \{(1,1), (2,1), (1,2), (2,2), (3,3)\}$.

Then prove R^{-1} S^{-1} is Symmetric.

$R \cup S = \{(1,1), (1,2), (2,1), (1,3), (3,1), (2,2), (3,3)\}$.

$R \cup S$ is Symmetric.

$R \cap S = \{(1,1), (1,2), (2,1)\}$.

$R \cap S$ is Symmetric.

Composition of two relation

Let $A = \{1,2,3,4\}$, $R = \{(1,2), (1,1), (1,3), (2,4), (3,2)\}$

& $S = \{(1,4), (1,3), (2,3), (3,1), (4,1)\}$. Find $S.R$.

$S.R = \cancel{\{(1,2), (2,1), (3,2), (4,}$

$\{(1,3), (1,1), (2,1), (3,3), (1,4)\}$.

Eg. Let $A = \{a,b,c\}$ & R & S are relation on A

$$M_R = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 1 \\ b & 1 & 1 & 1 \\ c & 0 & 1 & 0 \end{array} \qquad M_S = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 0 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \end{array}$$

find $M_{S.R}$

Sol

$M_R = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b)\}$.

$M_S = \{(a,a), (b,b), (b,c), (c,a), (c,c)\}$.

$M_{S.R} = \{(a,a), (b,a), (b,c), (b,b), (c,b), (c,c), (a,c)\}$

$$M_{S \cdot R} = \begin{array}{c|ccc} & a & b & c \\ \hline a & 1 & 0 & 1 \\ b & 1 & 1 & 1 \\ c & 0 & 1 & 1 \end{array}$$

$$M_{S \cdot R} = M_S \odot M_R.$$

$$M_{S \cdot R} = M_S \odot M_R.$$

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Theorem.

Let A, B and C be sets & R is a relation from A to B and S is a relation from B to C. Then $(S \cdot R)^{-1} = R^{-1} \cdot S^{-1}$

Proof:

$A \xrightarrow{R} B \xrightarrow{S} C$

Let $c \in C$, & $a \in A$.

If $(a, c) \in S \cdot R$

$\Leftrightarrow \quad (c, a) \in (S \cdot R)^{-1}$

If $b \in B$, with $(a, b) \in R$ & $(b, c) \in S$.

$\Leftrightarrow \quad (b, a) \in R^{-1}$ & $(c, b) \in S^{-1}$

$\Leftrightarrow \quad (c, a) \in R^{-1} \cdot S^{-1}$

$\therefore \quad (S \cdot R)^{-1} = R^{-1} \cdot S^{-1}.$

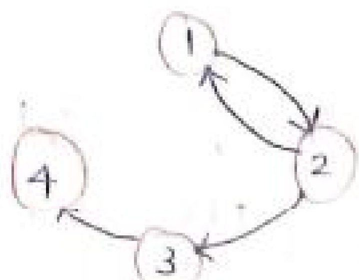
4.8 Transitive closure and warshall's Algorithm

Eg:

Let $A = \{1, 2, 3, 4\}$ and let $R = \{(1,2), (2,3), (3,4), (2,1)\}$

Find transitive closure of R

Sol:



Transitive closure,

$$R^{\infty} = \{(1,1), (1,3), (2,4), (1,2), (2,1), (2,2), (3,4), (2,3), (1,4)\}.$$

Find the transitive closure using warshall's Algorithm

$$M_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$(4,4), (4,5), (5,1), (5,5)\}$

Example Let $A = \{1, 2, 3, 4, 5\}$.

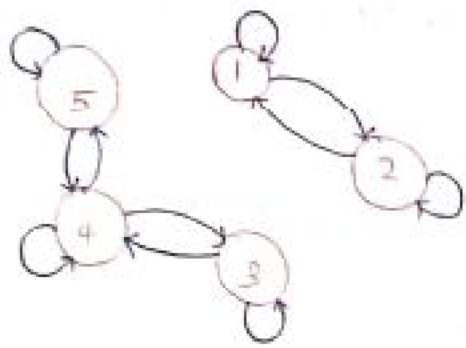
$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3)\}$
$(4,4), (5,5)$

$S = \{(1,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$

Find $(R \cup S)^{\infty}$ by warshall's algorithm.

Sol:

$R \cup S = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,1),$
$(4,5), (4,4), (5,4), (5,5)\}$.

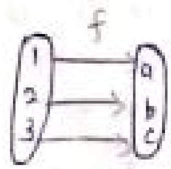


Warshall's Algorithm

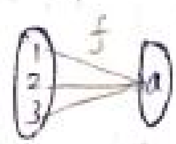
$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

$\therefore (R \cup S)^{\infty} = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4),$
$(5,3), (3,5), (4,3), (4,4), (4,5), (5,4), (5,$

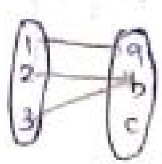
FUNCTION:



One to one



not a function becau
domain 1 has 2 ima



onto function.

Assignment problem :

3. Find transitive closure R^{∞} by warshall's Algorithm

$$M_R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ \& } A = \{1, 2, 3, 4\}.$$

UNIT - III

2/3/2020

Chapter - 5 : FUNCTIONS :

5.1 functions :

The origin of a function can be traced back to the great italian ~~philosopher~~ philosopher astronomer and mathematician Galileo Galilei observed the relationship between two variables.

Let A and B be non-empty sets a function f from A to B is denoted by $f : A \to B$ is a relation from A to B such that for all $a \in \text{Dom}(f)$, $f(a) \in \text{Range}(f)$ contains just one element of B.

Eg : Let $A = \{1, 2, 3, 4\}$ & $B = \{a, b, c, d\}$

Let $f(a) = \{(1, a), (2, a), (3, d), (4, c)\}$.

Then f is a function because

$f(1) = a$

$f(2) = a$

$f(3) = d$

$f(4) = c$

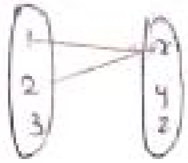
f is a function.

2. Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$. $R = \{(1, x), (2, x)\}$ &

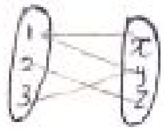
$S = \{(1, x), (1, y), (2, z), (3, y)\}$. Find whether R & S function :

Sol:

(i)



R is ~~...~~ a function whose range is $\{x\}$.

(ii) $S =$ 

S is not a function because $S(1) = x$, $S(1) = y$.

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Identity function:

Let A be any non empty set the identity function on A is defined by $I_A(a) = a$.

Composite function:

Let $f : A \to B$ and $g : B \to C$ be any two functions then the composite function $(g \cdot f)(x) = g(f(x))$

Eg:

Let $A = B = Z$ & C be any even integers. Let $f : A \to B$ & $g : B \to C$ be defined by

$$f(a) = a + 1$$

$$g(b) = 2b.$$ Find $g \cdot f$.

Sol:

$$(g \cdot f)(a) = g(f(a)) = g(a+1) = 2(a+1)$$

$$\therefore (g \cdot f)(a) = 2(a+1)$$

Special types of functions

Definitions:

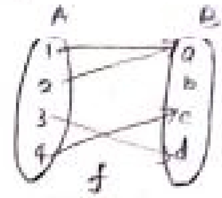
(i) Let f be a function from A to B then f is everywhere define if $Dom(f) = A$.

(ii) f is onto if $Ran(f) = B$.

(iii) f is one-to-one if $f(a) = f(a')$ iff $a = a'$.

Eg: 1. Consider the function $f = \{(1,a),(2,a),(3,d),(4,c)\}$ if $A = \{1,2,3,4\}$ & $B = \{a,b,c,d\}$.

Sol:



(i) $Dom(f) = A$

f is everywhere defined,

(ii) $Ran(f) \neq B$.

$\therefore f$ is not onto

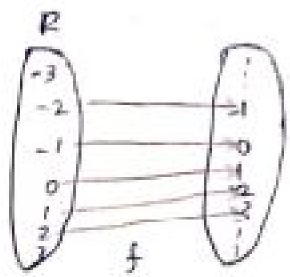
(iii) $f(1) = f(2) = a$

but $1 \neq 2$

f is not one-to-one

2. Let $A = B = \mathbb{Z}$, & $f : A \to B$ be defined by $f(a) = a+1$, $a \in A$ which of the special properties f posses.

Sol.:



(i) $Dom(f) = A$.

$\therefore f$ is everywhere defined

(ii) $Ran(f) = B$.

$\therefore f$ is onto.

(iii) $f(a) = f(b) \Rightarrow a+1 = b+1$

$a = b$

$\therefore f$ is $1-1$

Assignment prob:

1. $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$

$C = \{c_1, c_2\}$ $D = \{d_1, d_2, d_3, d_4\}$ & Consider 4 how functions from A to B & D to B & A to D & B to C.

(a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$.

$f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$.

$f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$.

$f_4 = \{(d_1, b_1), (d_2, b_2), (d_3, b_1)\}$.

determine whether these functions are everywhere define, 1-1 and onto.

Invertible functions: [Inventible function: f^{-1}]

A function $f: A \to B$ is said to be invertible if its inverse f^{-1} is also function.

Eg: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, $f = \{(1, a), (2, a), (3, d), (4, c)\}$ find whether f^{-1}[is function] exists.

Sol:

$f^{-1} = \{(a, 1), (a, 2), (d, 3), (c, 4)\}$.

$\therefore$ a has two images 1 & 2 $\therefore$ f^{-1} is not a function.

Theorem:

Let $f: A \to B$ be a function.

(a) Then f^{-1} is a function from B to A iff f is 1-1

If f^{-1} is a function then

(b) the function f^{-1} is 1-1

(c) f^{-1} is everywhere defined iff f is onto

(d) f^{-1} is onto iff f is everywhere defined.

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Proof: (a) If f^{-1} is not a function

if $b \in B$, $f^{-1}(b) = a_1 = a_2$ $\forall a_1, a_2 \in A$

$\Rightarrow b = f(a_1) = f(a_2)$

$\therefore f$ is not 1-1.

Conversely, suppose f is not 1-1

Then $f(a_1) = f(a_2) = b$ where $a_1, a_2 \in A$ & $b \in B$.

$\Rightarrow a_1 = a_2 = f^{-1}(b)$

$\therefore f^{-1}$ is not a function.

(b) If f is a fn. Then a function f^{-1} is also 1-1

Proof: $(f^{-1})^{-1} = f$ is a function.

$(f^{-1})^{-1}$ is a function

[by (a)] f^{-1} is 1-1

(c) f^{-1} is everywhere defined iff f is onto.

Proof:

Dom (f^{-}) = B = Ran (f).

$\therefore f^{-1}$ is every where defined

$\Leftrightarrow f$ is onto.

(d) f^{-1} is onto iff f is everywhere defined.

Proof: f^{-1} is onto

$\Leftrightarrow$ ran (f^{-1}) = A = dom (f)

f^{-1} is onto iff f is everywhere defined.

one-to-one Correspondence :

A function $f: A \to B$ is said to be one-to-one correspondence if f is one-to-one and onto and f^{-1} exists [f^{-1} is a function]

Eg:

Let R be the set of real #'s and let $f: R \to R$ be a function defined by $f(x) = x^2$, Is f invertible

Sol:

f is not 1-1

because $f(-3) = f(3) = 9$

$\therefore f^{-1}$ is not a function

i.e. f is not invertible

Theorem :

Let $f: A \to B$ be any function

Then (a) $I_B \circ f = f$

(b) $f \circ I_A = f$.

If f is 1-1 correspondence between A to B

Then (c) $f^{-1} \circ f = I_A$

(d) $f \circ f^{-1} = I_B$.

Proof :

(a) $(I_B \circ f)(a) = I_B(f(a))$

$= f(a)$

$\boxed{\therefore I_B \circ f = f}$

(b) $(f \circ I_A)(a) = f(I_A(a))$

$= f(a)$

$\boxed{\therefore f \circ I_A = f}$

(c) $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = I_A(a)$

$$\boxed{f^{-1} \circ f = I_A}.$$

(d) $(f \circ f^{-1})(a) = f(f^{-1}(a)) = I_B(a)$.

$$\boxed{f \circ f^{-1} = I_B.}$$

Theorem:

(a) Let $f : A \to B$ & $g : B \to A$ be a functions s.t $g \cdot f = I_A$ and $f \cdot g = I_B$. Then f is 1-1 correspondence between A to B. g is 1-1 correspondence between B to A and each is the inverse of the other.

(b) Let $f : A \to B$ & $g : B \to C$ be invertible. Then $g \cdot f$ is invertible & $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof:

Given: $g \cdot f = I_A$

$\therefore (g \cdot f)(a) = I_A(a) = a$.

i.e., $g(f(a)) = a$.

Also $f \cdot g = I_B$.

$(f \cdot g)(b) = I_B(b) = b$.

$\Rightarrow f(g(b)) = b$

If $f(a_2) = f(a_1)$.

$g(f(a_2)) = g(f(a_1))$.

$\Rightarrow a_2 = a_1$.

$\therefore f$ is 1-1

11-3.2020. Since $g(f(a)) = a$, $\forall a \in A$

$f(g(b)) = b$, $\forall b \in B$

$\therefore$ Dom $f = B$

and Dom $g = A$

$\therefore$ f and g are onto

$\therefore$ f and g are 1-1 correspondence.

$\therefore$ f^{-1} and g^{-1} exists.

$\Rightarrow$ g and f are invertible.

(b) Given $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

there (by (a)) f^{-1} and g^{-1} functions

$\therefore$ $g^{-1} \circ f^{-1}$ is also a function

$\therefore$ $(f \circ g)^{-1}$ is a function)

$\Rightarrow$ $f \circ g$ is invertible.

Eg: Let $A = B = R$, the set of real numbers, let $f: A \to B$ be defined as $f(x) = 2x^3 - 1$ and $g: B \to A$ be given by

$$g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$$

S.T f is bijection from A to B & g is bijection from B to A.

Proof: $f(x) = 2x^3 - 1$

Let $y = f(x)$

$y = 2x^3 - 1$

$\Rightarrow y + 1 = 2x^3$

$\frac{y}{2} + \frac{1}{2} = x^3$

$x = \sqrt[3]{\frac{y}{2} + \frac{1}{2}}$

$\therefore x = \sqrt[3]{\frac{y}{2} + \frac{1}{2}} = g(y)$

$\Rightarrow g(f(x)) = \sqrt[3]{y/_2 + 1/_2} = x.$

$\Rightarrow (g \cdot f)(x) = x.$

$g \cdot f = I_A$. g & f are bijection.

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5.2 Functions for Computer Science

Define: Characteristic function:

Let A be a subset of the universal set U the characteristic function of A is defined as

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Eg: If $A = \{4, 7, 9\}$. Find $f_A(7)$, $f_A(5)$, $f_A(1)$

Sol:

$f_A(7) = 1$ $7 \in A$

$f_A(5) = 0$ because $5 \notin A$

$f_A(1) = 0$ $1 \notin A$

Eg. Find the ceiling and flooring function of $f(1.5)$.

Sol: $f(1.5) = \lceil 1.5 \rceil = 2$. [Ceiling]

$f(1.5) = \lfloor 1.5 \rfloor = 1$ [Flooring]

Eg: Find the ceiling and flooring function of $f(-3)$, $f(-2.7)$

Sol:

i) Ceiling function $f(-3) = \lceil -3 \rceil = -3$

flooring function $f(-3) = \lfloor -3 \rfloor = -3$.

ii) Ceiling function $f(-2.7) = \lceil -2.7 \rceil = -2$

flooring function $f(-2.7) = \lfloor -2.7 \rfloor = -3$

5.3 Growth of functions:

Let f and g be functions we say that f is $O(g)$, denoted as $f = O(g)$ [f is big oh of g]. If there exist constant c and k s.t $|f(n)| \leq c\,|g(n)|$ for all $n \geq k$.

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Eg: S.T $f(n) = \frac{1}{2}n^3 + \frac{1}{2}n^2$ is $O(g)$ for $g(n) = n^3$. 12/3

Proof:

$$f(n) = \frac{1}{2}n^3 + \frac{1}{2}n^2$$

$$\le \frac{1}{2}n^3 + \frac{1}{2}n^3 \quad \text{if } n \ge 1$$

$$\therefore f(n) \le n^3 = 1(g(n))$$

where $c = 1$

$$\therefore f(n) \le c \cdot g(n)$$

$$\therefore f = O(g)$$

2. Let $f(x) = 3n^4 - 5n^2$ & $g(n) = n^4$. Then f & g have same order

Sol:

$$f(n) = 3n^4 - 5n^2$$

$$\le 3n^4 + 5n^4 \quad \forall n \ge 1$$

$$= 8n^4 = 8(g(n))$$

$$\therefore f(n) \le c\, g(x)$$

where $c = 8$.

$\therefore f$ is $O(g)$.

Now $g(n) = n^4 = 3n^4 - 2n^4 \le 3n^4 - 5n^2$ for $n \ge 2$.

$$g(n) \le 1 f(n)$$

$$\therefore g(n) \le c \cdot f(n)$$

$\therefore g$ is $O(f)$

$\therefore f$ & g have same order.

injection → one to one =

5.

12/3/2020. Big - Theta

We define a relation Θ big-Theta as $f \Theta g$. iff f & g have same order.

Theorem:

The relation (big Theta) Θ is an equivalence relation

Sol:

$f \Theta g \Rightarrow f = O(g)$ & $g = O(f)$ i.e, $f(n) \leq c_1 g(n)$ & $g(n) \leq c_2 g(n)$.

Now (i). $f(n) \leq c_1 g(n) \leq c_1 c_2 f(n)$.

$\therefore f(n) \leq c_3 f(n)$ (where $c_1 c_2 = c_3$)

$\therefore f = O(f)$

$\therefore \Theta$ is reflexive.

(ii) Since $f = O(g)$

& $g = O(f)$.

Θ is symmetric

(iii) If $f = O(g)$ & $g = O(h)$.

$\therefore f(n) \leq c_1 g(n)$

& $g(n) \leq c_2 h(n)$.

$f(n) \leq c_1 g(n) \leq c_1 c_2 h(n)$

$\Rightarrow f(n) \leq c_3 h(n)$ [$\because c_1 c_2 = c_3$]

$\therefore f = O(h)$

$\therefore \Theta$ is transitive

$\therefore \Theta$ is an equivalence relation.

5.4 Permutation Function:

A bijection from a set A on itself is a permutation of A.

Eg:

Let $A = \{1, 2, 3\}$. Then find the all permutations of A & find P_4^{-1} & $P_2 \circ P_3$.

Sol:

$$I_n = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

$$P_4^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$P_2 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$P_2 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Eg:

Let $A = \{1, 2, 3, 4, 5\}$ find the cycle permutation of $(1, 3, 5)$

Sol:

$$(1, 3, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$$

Eg: Let $A = \{1, 2, 3, 4, 5, 6\}$ Compute $(4, 1, 3, 5) \circ (5, 6, 3)$ & $(5, 6, 3) \circ (4, 1, 3, 5)$.

Sol:

$$(4, 1, 3, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$$

$$(5, 6, 3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$$

$(4,1,3,5)\cdot(5,6,3) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}$$

$(5,6,3)\cdot(4,1,3,5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 4 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix}$$

(1,4,2) = 3 length
(4,1,3,5) = 4 length

Even (or) odd Permutation:

A Cycle of Length 2 is transposition.

Eg: write the permutation $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$

(1,3,6), (2,4,5) (7,8)

as a product of transposition.

Sol:

$P = (1,3,6),(2,4,5),(7,8)$

$= (7,8),(2,5),(2,4),(1,6),(1,3)$

$\therefore$ P is odd permutation (5 transposition)

UNIT - 4.

Chapter - 6: Order Relation & Structures

6.1 Partially Ordered Set (Poset)

Partially ordered set definition

A relation R on a set A is called partially order if R is reflexive, Antisymmetric and transitive, then R is called Partially ordered relation and A is called poset.

Let A be a Collection of subset of set S the relation subset is a partially ordered on A. So $(A, \subseteq)$ is a Po set

Eg:

Let Z^+ be the set of all +ve integers then the relation $\leq$ is a partial order on Z^+ then $(Z^+, \leq)$ is a poset.

Theorem:

If $(A, \leq)$ & $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset, with partial order $\leq$ denoted by $(a, b) \leq (a', b')$. If $a \leq a' \in A$ & $b \leq b' \in B$

Proof:

(i) $(A, \leq)$ & $(B, \leq)$ are posets.

$\therefore a \leq a$ & $b \leq b$ $\forall a \in A$ & $b \in B$.

$\Rightarrow (a, b) \leq (a, b)$

$\therefore \leq$ is a reflexive on $A \times B$

(ii) Also $(A, \leq)$ is a symmetric

Then $a \le a'$

& $a' \le a$ where $a, a' \in A$.

$\Rightarrow a = a'$

$(A, \le)$ is a Antisymmetric.

$b \le b'$

$b' \le b \rightarrow b = b'$, where $b, b' \in B$.

Now $(a, b) \le (a', b')$

& $(a', b') \le (a, b)$

$\therefore (a, b) = (a', b')$

$\therefore \le$ is an Antisymmetric on $A \times B$.

(iii) $(A, \le)$ & $(B, \le)$ are posets

i.e., if $a \le a'$ & $a' \le a''$ of $a, a', a'' \in A$.

$\Rightarrow a \le a''$

If $b \le b'$ & $b' \le b''$

$\Rightarrow b \le b''$ where $b, b', b'' \in B$

If $(a, b) \le (a', b')$

& $(a', b') \le (a'', b'')$

Now $(a, b) \le (a'', b'')$.

$\therefore \le$ is a transitive

$\therefore (A \times B, \le)$ is a poset.

3/2020

Hasse Diagram:

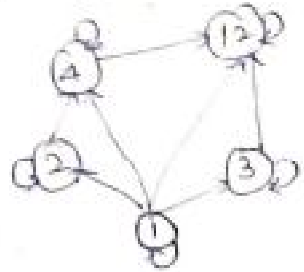
Let $A = \{1, 2, 3, 4, 12\}$

Consider the partial order of divisibility on A

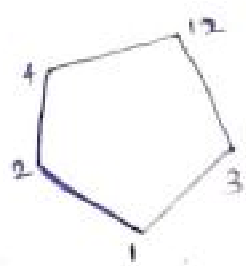
Draw A Hasse diagram of the poset $(A, \le)$.

Sol:

Diagraph of $(A, \leq)$ is



Hasse Diagram is



Eg: Let $S = \{a, b, c\}$ and $A = P(S)$.
draw Hasse diagram for the poset A with partial order $\subseteq$

Sol:

$$A = P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$$

Hasse diagram:

$\{a,b,c\}$

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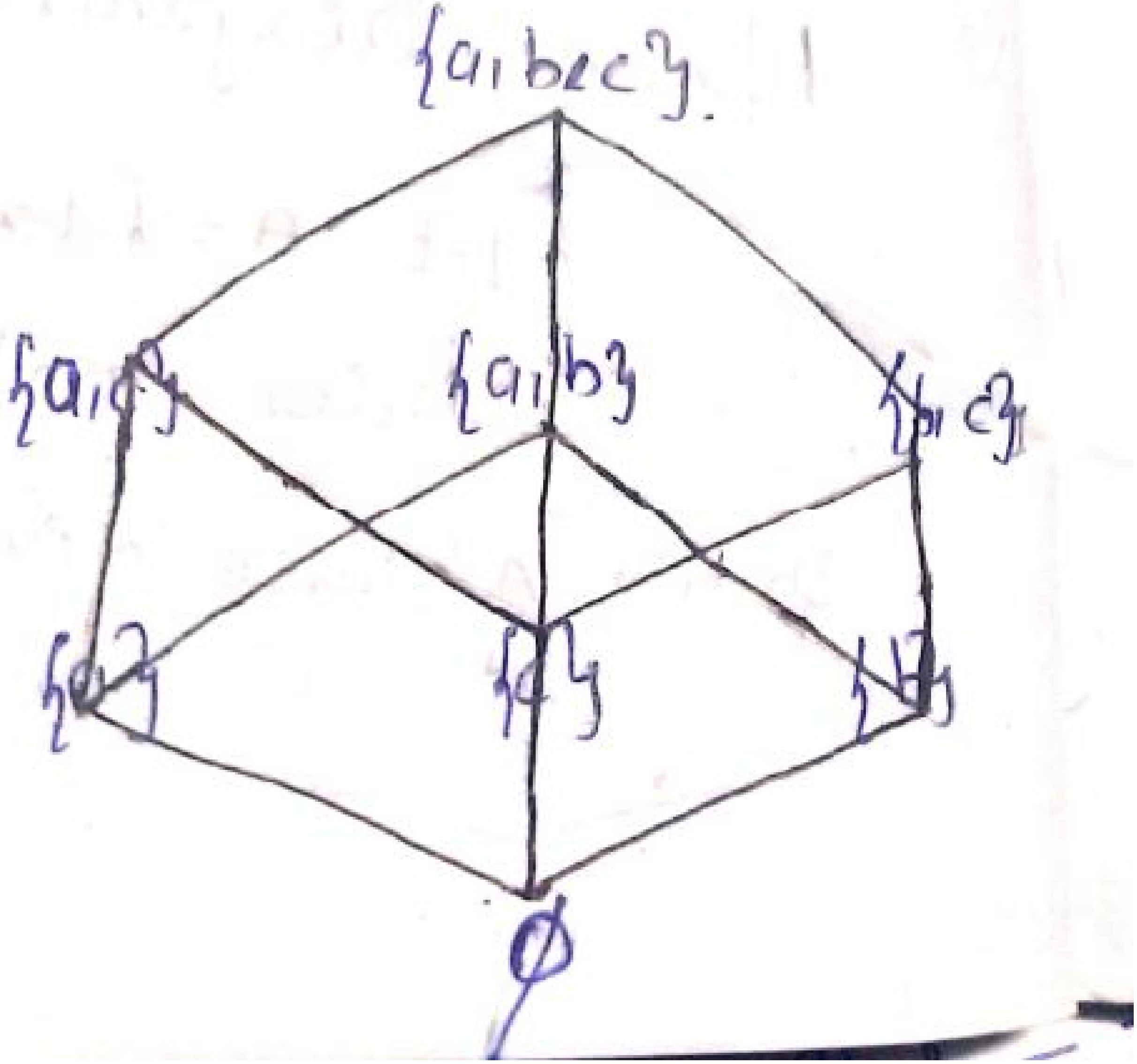

{a,b,c}
{a,b}
∅

Isomorphism :

Let $(A, \leq)$ & $(A', \leq')$ be posets let 'f' from A to A' be one to one correspondence between A and A' then the function f is called an isomorphism from $(A, \leq)$ to $(A', \leq')$.

If for any a, b in A, $a \leq b$ iff $f(a) \leq' f(b)$ then f is said to be an isomorphism.

Eg: Let A be the set of positive integers Z^+ and $\leq$ be the usual partial order on A. Let A' be the set of positive even integers and $\leq'$ be the usual partial order on A', then the function 'f' from A to A' be given by $f(a) = 2a$. is an isomorphism from $(A, \leq)$ to $(A', \leq')$

Solution:

Given $f(a) = 2a$.

(i) If $a = b$

$\Leftrightarrow 2a = 2b$

$\Leftrightarrow f(a) = f(b)$

$\therefore f$ is 1-1

(ii) If $c = 2a$ where $c \in A'$

$\Rightarrow a = c/2$ for every $a \in A$.

$\therefore f$ is onto.

(iii) If $a \leq b$.

$\Rightarrow 2a \leq 2b$

$\Rightarrow f(a) \leq f(b)$

$\therefore f$ is a homomorphism.

$\therefore f$ is 1-1 onto homomorphism.

$\therefore f$ is an isomorphism.

Principle of Correspondence.

Theorem :

If the elements of B have any property relating to one another or the other elements of A and this property can be defined entirely in the relation $\leq$, then the elements of B' must posses the same property defined in $\leq'$.

Maximal element :

Defn: (i) An element $a \in A$ is called a maximal element of A if there is no element $c \in A$ s.t $a < c$.

Minimal element :

(ii) An element $b \in A$ is called minimal element of A if there exists no element $c \in A$ s.t $c < b$

Theorem :

Let A be a finite non-empty poset with partial order $\leq$. Then A has atleast one maximal element and atleast one minimal element.

Proof :

Let 'a' be any element of A. If 'a' is not maximal, we can find an element $a_1 \in A$ s.t $a < a_1$. If a_1 is not maximal, we can find $a_2 \in A$ s.t $a_1 < a_2$. We continue like this, we can find $a < a_1 < a_2 \ldots < a_{k-1} < a_k$.

$\therefore$ we canot find $a_k < b$ for any $b \in A$. a_k is maximal element of $(A, \leq)$.

Illy we can find minimal element of $(A, \leq)$

Greatest element :

An element $a \in A$ is called a greatest element of A if $x \leq a$ for all $x \in A$.

Least element :

An element $a \in A$ is called least element of A if $a \leq x$ for all $x \in A$.

Example:

Let A be the poset of non-negative real number with usual partial order $\leq$. Then 0 is the minimal element of A. There is no maximal element.

Example:

Let $S = \{a, b, c\}$. $A = P(S)$ be the collection of subset of S.

Then ϕ is the least element of A and S is the greatest element of A.

Theorem:

A poset has at most one greatest element and atmost one least element.

Proof:

Suppose that a and b are greatest element of a poset A. Then since b is a greatest element, we have $a \leq b$.

Similarly since a is a greatest element we have $b \leq a$. $\therefore a = b$ by antisymmetric property.

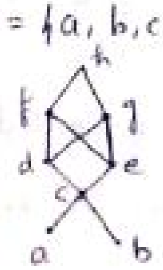
$\therefore$ It has one maximal element and one minimal element.

Unit element & Zero element:

The greatest element of a poset is denoted by 'I' and it is called unit element. The least element of poset is '0' and it is called zero element.

Example:

Consider the poset $A = \{a, b, c, d, e, f, g, h\}$ whose Hasse diagram is



Find the upper and lower bound of $B_1 = \{a, b\}$ & $B_2 = \{c, d, e\}$.

Solution:

(a) B_1 has no lower bound. Upper bounds are c, d, e, f, g, h.

(b) The upper bounds of B_2 are $f, g, \& h$.

Lower bounds of B_2 are $c, a \& b$.

Upper bound and Lower bound:

Consider a poset A and a subset B of A. An element $a \in A$ is called an upper bound of B if $b \leq a \ \forall \ b \in B$.

An element $a \in A$ is called lower bound of, if $a < b \ \forall \ b \in A$

Least upper bound:

Let A be a poset and B a subset of A. An element $a \in A$ is called least upper bound, of B if a is an upper bound of B and $a \leq a'$ whenever a' is an upper bound of B. Thus $a = LUB(B)$ if $b \leq a$ for all $b \in B$ and if whenever $a' \in A$ is also an upper bound of B then $a \leq a'$

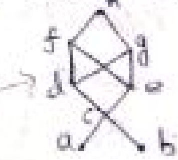
Greatest lower bound:

An element $a \in A$ is called a greatest lower bound of B, if a is the lower bound of B and $a' \leq$ whenever a' is a lower bound of B. Thus $a = GLB(B)$ if $a \leq b$ for all, $b \in B$ whenever $a' \in A$ is also a lower bound of B, Then $a' \leq a$.

Example:

Let $A = \{a, b, c, d, e, f, g, h\}$ with subsets $B_1 = \{a, b\}$ & $B_2 = \{c, d, e\}$. Find all LUBs and GLB of B_1 & B_2.

Solution: Hasse diagram is $\rightarrow$



Solution :

(a) B_1 has no lower bound, it has no GLB.

But LUB (B_1) = c

(b) Since lower bound of B_2 are ~~…~~ c, and b.

$\therefore$ GLB (B_2) = c

The upper bounds of B_2 are f, g and h

f and g are not comparable

$\therefore B_2$ has no LUB.

Example :

Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} be a poset whose Hasse diagram is

Find LUB & GLB of B = {6, 7, 10}.

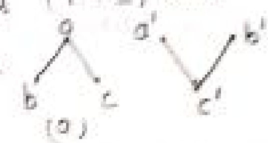
Solution :

LUB (B) = 10

GLB (B) = 4.

Example :

S.T the posets (A, ≤) and (A', ≤') whose Hasse diagrams are



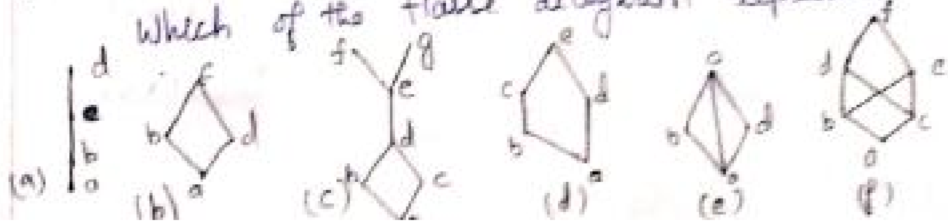
are not isomorphic.

Lattice :

A lattice is a poset (L, ≤) in which every subset {a, b} consisting of two elements has a least upper bound and greatest lower bound. We denote GLB (a, b) = a ∧ b call it us meet of a & b.

Example :

Which of the Hasse diagram represent lattice :



Solution :

Hasse diagrams (a), (b), (d) & (e) represent lattices. (c) does not represent lattice because $f \vee g$ does not exist. (f) doesn't represent a lattice because neither $d \wedge e$ nor $b \vee e$ exist.

Example :

Let S be a set and L ; P(S). Then $(L, \subseteq)$ is a lattice

Theorem :

If $(L_1, \leq)$ and $(L_2, \leq)$ are lattices then $(L, \leq)$ is a lattice where $L = L_1 \times L_2$.

Proof :

Join of L_1 is $\vee_1$ and meet of L_1 is $\wedge_1$

Join of L_2 is $\vee_2$ and meet of L_2 is $\wedge_2$.

Since L is a poset. If $[a_1, b_1]$ & $[a_2, b_2] \in L$

Then $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2) \in L$

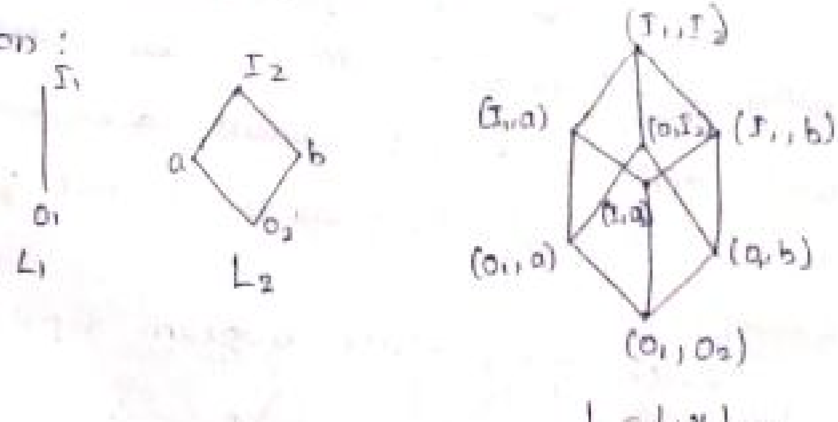
$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2) \in L$.

$\therefore$ L is a lattice

Example :

If L_1 & L_2 are lattices shown in figure (a) & (b). Then $L_1 \times L_2$ is a lattice

Solution :



$L = L_1 \times L_2$.

Isomorphic lattice :

If $f : L_1 \to L_2$ be an isomorphism from posed $(L_1, \leq_1)$ to the poset $(L_2, \leq_2)$ and if a & b are element of L_1. Then $f(a \wedge b) = f(a) \wedge f(b)$ & $f(a \vee b) = f(a) \vee f(b)$. Then L_1 & L_2 are isomorphic lattice.

Properties of lattice :

(i) $a \leq a \vee b$ and $b \leq a \vee b$, $a \vee b$ is an upper bound of a and b.

(ii) If $a \leq c$ & $b \leq c$ then $a \vee b \leq c$; $a \vee b$ is the upper bound of a and b.

(iii) $a \wedge b \leq a$ & $a \wedge b \leq b$; $a \wedge b$ is a lower bound of a and b.

(iv) If $c \leq a$ and $c \leq b$ then $c \leq a \wedge b$; $a \wedge b$ is the gLB of a and b.

Theorem :

Let L be a lattice. Then for every a and b in L. (a) $a \vee b = b$ iff $a \leq b$.

(b) $a \wedge b = a$ iff $a \leq b$

(c) $a \wedge b = a$ iff $a \vee b = b$.

Proof :

(a) Suppose that $a \vee b = b$. Since ~~a ≤ a∨b~~ $a \leq a \vee b = b$ we have $a \leq b$. Conversely, if $a \leq b$, then since $b \leq b$, b is an upper bound of a and b

$\therefore a \vee b \leq b$. Since $a \vee b$ is an upper bound $b \leq a \vee b$, so $a \vee b = b$.

(b) Suppose $a \wedge b = b$. Since $a \leq a \wedge b = b$.

we have $a \leq b$.

Conversely if $a \leq b$.

Since $a \leq a$ a is an lower bound of a and b

$\therefore a \leq a \wedge b$.

Since $a \wedge b$ is an lower bound

$a \wedge b \leq a$

$\therefore a = a \wedge b$.

(c) If $a \wedge b = a$

$\Leftrightarrow a \leq b$ (by b)

$\Leftrightarrow a \vee b = b$ (by a)

If $a \vee b = b$

$\Leftrightarrow a \leq b$ (by a)

$\Leftrightarrow a \wedge b = a$ (by b)

Distributive lattice :

A lattice is called distributive if for any elements a, b and c in L have the following properties

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Complem :

A element $a' \in L$ is called Complem of a if $a \vee a' = I$ & $a \wedge a' = 0$.

Example :

Let T be the set of all even integers. S.T the semigroups $(Z, +)$ and $(T, +)$ are isomorphic.

Solution :

Define $f : Z \to T$ by $f(a) = 2a$

If $f(a_1) = f(a_2)$

$2a_1 = 2a_2$

$a_1 = a_2$

$\therefore f$ is 1-1

Suppose b is any even integer in T

Then $b = 2a$

$$f(a) = f(b/2) = 2(b/2) = b$$

$\therefore$ f is onto.

$$f(a+b) = 2(a+b) = 2a + 2b = f(a) + f(b)$$

$\therefore$ f is an isomorphism from Z to T

Let $S = \{a, b, c\}$ and $T = \{x, y, z\}$. The semi group structure of S & T are

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$*$	x	y	z
x	z	x	y
y	x	y	z
z	y	z	x

Let $f(a) = y$, $f(b) = x$ & $f(c) = z$. Then S and T are isomorphic.

Theorem:

Let L be a bounded distributive lattice. If a complement exists, it is unique.

Proof:

Let a' & a'' be complements of the element $a \in L$.

Then $a \vee a' = I$, $a \vee a'' = I$

$a \wedge a' = 0$, $a \wedge a'' = 0$.

Using distributive laws,

$$a' = a' \vee 0 = a' \vee (a \wedge a'')$$

$$= (a' \vee a) \wedge (a' \vee a'')$$

$$= I \wedge (a' \vee a'') = a' \vee a''.$$

Also $a'' = a'' \vee 0 = a'' \vee (a \wedge a')$

$$= (a'' \vee a) \wedge (a'' \vee a')$$

$$= I \wedge (a' \vee a'') = a' \vee a''.$$

Hence $a' = a''$.

UNIT - V

Capter 9:

Semi Groups and groups

Binary operation:

A binary operation on a set A is an everywhere defined function $f: A \times A \to A$.

Example:

Let $A = Z$, Define $a * b$ as $a+b$. Then $*$ is an binary operation on Z.

Semi group:

A semi group is a nonempty set S together with an associative binary operation $*$ defined on S

Examples:

(i) $(Z,+)$ is a commutative semi group

(ii) The number 0 is an identity is the semi group $(Z,+)$.

(iii) The semigroup $(Z^+,+)$ has no identity element.

Monoid:

A monoid is a semigroup $(S,*)$ that has an identity

Sub-semi group and Sub-monoid:

Let $(S,*)$ be a semigroup and let T be a subset of S. If T is closed under the operation $*$, then $(T,*)$ is called a subsemigroup of $(S,*)$. Let $(S,*)$ be a monoid with identity e and let T be a non-empty subset of S. If T is closed under the operation * and $e \in T$, then $(T,*)$ is called a Submonoid of $(S,*)$.

Isomorphism:

Let $(S,*)$ and $(T,*')$ are two semigroups. A function $f: S \to T$ is called an isomorphism from $(S,*)$ to $(T,*')$ if it is a 1-1 correspondence from $(S,*)$ to $(T,*')$ if it is a 1-1 correspondence from S to T and if $f(a*b) = f(a) *' f(b)$ for all a and b in S.

Homomorphism:

Let $(S,*)$ and $(T,*')$ be two semigroups. A every where defined function $f: S \to T$ is called a homomorphism from $(S,*)$ to $(T,*')$ if $f(a*b) = f(a) *' f(b)$ for all a and b in S.

Theorem:

Let $(S,*)$ and $(T,*')$ be monoids with identities e and e'. Let $f: S \to T$ be a homomorphism from $(S,*)$ onto $(T,*')$. Then $f(e) = e'$.

Theorem:

Let f be an homomorphism from a semi group $(S,*)$ to a semigroup $(T,*')$. If S' is a ~~subgroup~~ subsemigroup of $(S,*)$, then

$f(S') = \{t \in T \mid t = f(s)$ for some $s \in S'\}$ the image of S under f is a subsemigroup of $(T,*')$

Proof:

If t_1 and t_2 be any elements of $f(S')$ then there exists S_1 and $S_2 \in S'$ with $t_1 = f(S_1)$ & $t_2 = f(S_2)$.

Then $t_1 *' t_2 = f(s_1) *' f(s_2) = f(s_1 * s_2)$

$= f(s_3)$

$\therefore t_1 *' t_2 \in f(s')$

$\therefore f(s')$ is closed under $*$ $\therefore f(s')$ is closed under associative. $\therefore f(s')$ is a subsemigroup $(T, *')$

Theorem:

If f is homomorphism from a commutative semigroup $(S, *)$ onto a semigroup $(T, *')$.

Then $(T, *')$ is also commutative

Proof:

Let t_1 and t_2 be any elements of T. Then there exists s_1 and s_2 in S with $t_1 = f(s_1)$ and $t_2 = f(s_2)$

$\therefore t_1 *' t_2 = f(s_1) *' f(s_2) = f(s_1 * s_2)$

$= f(s_2 * s_1) = f(s_2) *' f(s_1)$

$= t_2 *' t_1$

$\therefore (T, *')$ is commutative.

Congruence relation:

An equivalence relation R on the semigroup $(S, *)$ is called a congruence relation if $a R a'$ and $b R b' \Rightarrow (a * b) R (a' * b')$.

Example:

Consider the semigroup $(Z, +)$ and the equivalence relation R on Z defined by $a R b$ iff $a \equiv b \pmod 2$.

Solution

If $a \equiv b \pmod 2$ and $c \equiv d \pmod 2$

Then $2 \mid a-b$ & $2 \mid c-d$.

$\therefore$ $a-b = 2m$ & $c-d = 2n$ where $m, n \in \mathbb{Z}$.

Now $(a-c)+(c-d) = 2m+2n = 2(m+n)$

$\therefore$ $(a+c) = (b+d) \pmod 2$

$\therefore$ relation is an Congruence relation

Example:

Consider the semigroup $(\mathbb{Z}, +)$ when '+' is ordinary addition. Let $f(x) = x^2 - x - 2$. Define aRb iff $f(a) = f(b)$. Then R is not an equivalence relation

Solution:

$-1 R 2$ since $f(-1) = f(2) = 0$.

$-2 R 3$ since $f(-2) = f(3) = 4$.

but $((-1)+(-2)) \not R$ $2+3$.

because $f(-3) = 10$ & $f(5) = 18$.

Theorem:

(Fundamental Homomorphism theorem:)

Let $f: S \to T$ be a homomorphism of the semigroup $(S, *)$ onto the semigroup $(T, *)$. Let R be the relation on S defined by aRb if and only if $f(a) = f(b)$ for a and b in S. Then

(a) R is a Congruence relation

(b) $(T, *')$ and the quotient semigroup $(S/R, \oplus)$ is isomorphic.

Proof:

To prove R is an equivalence relation if aRa for every $a \in S$.

Since $f(a) = f(a)$

If aRb then $f(a) = f(b)$

$f(b) = f(a) \Rightarrow bRa$.

If aRb & bRc, Then $f(a) = f(b)$ & $f(b) = f(c)$

$f(a) = f(c)$ aRc

$\therefore$ R is an equivalence relation.

~~Theorem~~:

(a) Now $f(a) = f(a_1)$ & $f(b) = f(b_1)$

$f(a) *' f(b) = f(a_1) *' f(b_1)$

Since f is homomorphism

$f(a * b) = f(a_1 * b_1)$

$\therefore (a * b) R (a_1 * b_1)$

$\therefore$ ~~$(a * b$~~

$\therefore$ R is an congruence relation.

(b) Consider $\bar{f} : S/R \to T$ defined by

$\bar{f} = \{([a], f(a)) \mid [a] \in S/R\}$.

Suppose $[a] = [a']$ Then aRa'

$\therefore f(a) = f(a')$

$\therefore \bar{f}$ is a function.

$\therefore \bar{f}[a] = f(a)$ for $[a] \in S/R$.

To prove f is one to one

$\bar{f}([a]) = \bar{f}([a'])$

$f(a) = f(a')$

$[a] = [a']$

$\therefore f$ is 1-1

Next suppose $b \in T$ since f is onto

$f(a) = b$ for some $a \in S$.

$\therefore \bar{f}([a]) = f(a) = b$

$\therefore \bar{f}$ is onto.

Next $\bar{f}([a] * [b]) = \bar{f}([a*b]) = f(a*b) = f(a) *' f(b)$

$= (\bar{f}[a]) *' (\bar{f}[b])$

$\therefore \bar{f}$ is homomorphism

Groups:

A group $(G, *)$ is a monoid with identity e.

Abelian:

A group G is Abelian if $ab = ba \; \forall \; a$ and b.

Examples (i):

The set of all integers Z with addition is an Abelian group.

(ii) A set of all non-zero real numbers under multiplication is a group.

(iii) Let G be the set of all non-zero real numbers and let $a*b = ab/2$. Show that $(G, *)$ is an abelian group.

Proof: (1) $a*b = \frac{ab}{2} \in G$.

(2) $a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{abc}{4}$

& $(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{abc}{4}$.

$\therefore a*(b*c) = (a*b)*c \in G$.

$\therefore G$ is closed under associative property

(3) $a * 2 = \frac{(a)(2)}{2} = a = 2 * a = \frac{2(a)}{2}$

$\therefore$ 2 is the Identity element in G

(4) If $a \in G$ then if $a' = 4/a$.

$a * a' = a * \frac{4}{a} = \frac{4a}{2a} = 2$

$\therefore a * a' = a' * a = e$

$\therefore a' = 4/a$ is the inverse of a.

~~$\therefore a * a' = a' * a = e$.~~

Since $a * b = b * a$ for all a and b in G

$\therefore$ G is an abelian group.

Theorem:

Let G be a group and let a, b and c be elements of G. Then

(a) $ab = ac \Rightarrow b = c$ (left . c. L)

(b) $ba = ca \Rightarrow b = c$ (R.c.L)

Proof:

Suppose $ab = ac$

Multiplying both sides by a^{-1} on left

(a) $a^{-1}(ab) = a^{-1}(ac)$

$(a^{-1}a)b = (a^{-1}a)c$

$\Rightarrow eb = ec$

$\Rightarrow b = c$

(b) $ba = ca$.

Multiple a^{-1} by right side on both sides

$(ba)a^{-1} = (ca)a^{-1}$

$b(aa^{-1}) = c(aa^{-1}) \Rightarrow b = c$

Theorem.

Let G be a group and $a \in G$.

Define a function $M_a : G \to G$ by the formula

$M_a(g) = ag$. Then M_a is 1-1.

Proof:

$M_a(g_1) = M_a(g_2)$

$ag_1 = ag_2$

$g_1 = g_2$ (by L.C.L.)

$\therefore M_a$ is 1-1

Theorem:

Let G be a group and let a and b be elements of G. Then

(a) $(a^{-1})^{-1} = a$.

(b) $(ab)^{-1} = b^{-1}a^{-1}$

Proof:

(a) since $a^{-1}a = aa^{-1} = e$.

a^{-1} is the inverse of a.

$\therefore (a^{-1})^{-1} = a$.

(b) $(ab)(b^{-1}a^{-1}) = a(bb^{-1}a^{-1}) = a(ea^{-1})$

$a(a^{-1}) = e$

$\therefore b^{-1}a^{-1} = (ab)^{-1}$

Theorem:

Let G be a group and let a and b elements of G. Then

(a) The equation $ax = b$ has a unique solution in G.

(b) The equation $ya = b$ has a unit solution in G.

Proof:

(a) The element $x = a^{-1}b$ is a solution of $ax = b$. since $a(a^{-1}b) = (aa^{-1})b = eb = b$.

~~Suppose~~ $\cancel{a(a^{-1}b) = (aa^{-1})b = eb = b}$.

Suppose x_1 & x_2 are solutions of $ax = b$.

$$ax_1 = b \text{ \& } ax_2 = b$$

$$ax_1 = ax_2$$

$$x_1 = x_2$$

(b) Suppose y_1 & y_2 are solutions of $ya = b$

$\Rightarrow y_1 a = b$ & $y_2 a = b$

$\Rightarrow y_1 a = y_2 a$

$y_1 = y_2$.

$\therefore ya = b$ has a unique solution.